Mode Decomposition and Fourier Analysis of Physical Fields in Homogeneous Cosmology

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In this work the methods of mode decomposition and Fourier analysis of quantum fields on curved spacetimes previously available mainly for the scalar fields on Friedman-Robertson-Walker spacetimes are extended to arbitrary vector fields on general spatially homogeneous spacetimes. This is done by developing a rigorous unified framework which incorporates mode decomposition, harmonic analysis and Fourier analysis. Explicit constructions are performed for a variety of situations arising in homogeneous cosmology. A number of results concerning classical and quantum fields known for very restricted situations are generalized to cover almost all cosmological models.

Contents

Introduction

While there is no generally accepted and observationally predictive quantum theory of gravity to date, there are a number of approaches or partial solutions to the description of the interaction of quantum fields and gravity including quantum field theory on curved spacetimes, non-commutative spaces, string theory etc. The current work pertains to the first approach which is an adequate and consistent theoretical framework for astronomy and cosmology. The basic idea of quantum field theory in curved spacetimes (QFT in CST) is: Gravity is described in the framework of (non-quantized) general relativity, while matter fields propagating in a gravitational background are described as quantized fields. The gravitational back-reaction induced by the quantized matter fields is modelled by the semiclassical Einstein equation

$$G_{\mu\nu} + g_{\mu\nu}\Lambda = 8\pi\langle T_{\mu\nu}\rangle_{\omega},$$

where on the left hand side are the Einstein tensor plus cosmological term, and on the right hand side the expectation value of the stress energy tensor of the matter fields in some suitable state ω. It is believed that this approach is capable of describing quantitatively a wide range of situations where spacetime curvatures are very high, such as early cosmology before or during inflation, or the vicinity of black holes.

This theory has encountered a remarkable success in describing several phenomena that are inherent to the physical situations for which it is designed for, e.g., the Hawking radiation, the Unruh effect and the cosmological particle creation. As cosmology is the the main dedication of the current work the latter effect will be our primary physical target. The stony way from the axioms of the quantum field theory to the rigorous and at the same time explicit description of the cosmological particle creation proceeds through a whole bunch of refined mathematical methods. This task has been first accomplished recently in [14]. for a particular idealized model of the field and of the spacetime. Namely, a Friedman-Robertson-Walker(FRW) spacetime has been considered, i.e., a model where the universe is assumed to be homogeneous and isotropic at large scale. The particle creation rate has been calculated both formally and numerically in the homogeneous and isotropic states of low energy of the scalar Klein-Gordon field under additional mild continuity conditions. The final step of a theoretical investigation is the comparison of the predictions with the observational data. In principle the pattern of the microwave background radiation(CMB) measured by WMAP and Planck can serve as observational data for comparison with the theoretical picture of the early epoch of the universe. However, to really describe a cosmological phenomenon one has to generalize the results to higher dimensional vector fields (e.g., Maxwell, Dirac). Furthermore, as the first CMB data already indicated the assumed isotropy of the universe at large scale may be disputable. It may happen that the analysis of newly acquired data from the Planck mission force one to consider models with a homogeneous but anisotropic universe. Therefore there is also a need to extend the calculations from FRW to purely homogeneous (Bianchi) or only partially isotropic (LRS) spacetimes. It is the aim of the present thesis to develop a unified and coherent framework of mathematical methods which extend those used before in particular cases to a generality which covers practically all realistic cosmological situations. More explicitly, we unify the machineries of mode decomposition, harmonic analysis and Fourier analysis as applied to the cosmological situations.

We proceed to a more mathematical discussion of these methods. A primary tool for obtaining explicit constructions are geometric symmetries. One of the merits of geometric symmetries is the possibility of the separation of variables in the field equation which helps to obtain explicit solutions. The mode decomposition of the solutions of the field equation (also referred to as the Fourier method in PDE, or expansion into harmonic oscillators in physics) was probably first applied in the cosmological context by Parker [44] who performed it on the flat FRW spacetime. The idea of the method is that one tries to separate the time variable in the field equation, and looks for solutions as linear combinations of products $X(\vec{x})T(t)$ where X depends only on the spatial coordinates and T only on time. What Parker discovered is that this is possible on FRW spacetimes and represents a very handy tool for the analysis of the dynamics. Many authors concentrate on FRW situation because the mode decomposition is basically developed only for this case.

A thorough analytical investigation of the method in the cosmological context was conducted in [22], where an abstract functional analytical eigenfunction expansion was introduced as a methodological background. The theory of the method does not seem to have been developed any further until nowadays. In particular, the following questions remain open. What are the precise limits of applicability of the mode decomposition by means of separation of the time variable? How many different decompositions are possible for the same geometrical setup? When and how can the decomposition be extended to weak (distributional) solutions of the field equations? These questions are given satisfactory answers in the present thesis.

First our geometrical setup is introduced which is basically an n-dimensional complex vector bundle $\mathcal{T}$ over a 4-dimensional globally hyperbolic Lorentzian manifold M together with a pseudo-Riemannian fiber metric $\mathfrak{g}$ and a connection wave operator $\Box^{\nabla} + m^{\star}$, where ∇ is a metric linear connection on $\mathcal{T}$ and $m^{\star}$ is a variable mass term which possibly includes the curvature coupling term. Spinor fields such as Dirac field can be realized as such vector bundles with additional spinor structure. The Maxwell field can also be cast into a hyperbolic field under topological constraints on the spacetime M [18],[20],[2]. In general the covariant quantization of the Maxwell field is problematic [12]. Then we present the spectral mode decomposition in rigged Hilbert spaces as developed by [43],[37], which is the most abstract and general form of the spectral Fourier transform. Section 1.3 contains one of the main results of the chapter. First a precise definition is given of what we mean by a mode decomposition by the separation of the time variable, and then in proposition 3 it is shown that under the stated geometrical circumstances the mode decomposition is always possible, and, moreover, this is the only way such a mode decomposition can arise. The ensuing mode equations are described explicitly and a number of consequences of the assumed conditions are presented. Then the time dependent Fourier transform is defined and its properties are analyzed. In particular we use the energy estimates for the time dependent harmonic oscillator (obtained in the appendix) to show how the magnitudes of the mode solutions can be uniformly estimated by their initial data at time 0. Even stronger results are obtained for the case that the bundle is analytic. Section 1.5 presents the second major result of the chapter. It is shown that if a certain functional analytical condition (Eq.1.19) holds then then the mode decomposition can be extended to distributional solutions of the field equation. Finally, Section 1.6 deals with the propagator E of the field. The newly developed mode decomposition technique is applied to obtain a fairly natural form for E which generalizes that obtained in [32].

Another advantage of geometric symmetries is the possibility to apply harmonic analysis. This is particularly true for the cosmological models where a rather rich group of spatial isometries is imposed. Then the spatial sections of the spacetime can be considered as homogeneous spaces, and the spatial Fourier transform can be investigated in much more detail with many explicit consequences. These properties then can be dragged to the time dependent Fourier transform and hence to the mode decomposition. The harmonic analysis of FRW symmetry groups is well known since long, but as discussed above isotropy is not as fundamental in cosmology as homogeneity, and one is also interested in cosmological models which are anisotropic. The isometry groups of these spaces are described by Bianchi groups with their quotients and semidirect extensions (in case of

LRS models). Some of these groups are solvable, others are semisimple, with finite or infinite center. Therefore it is not easy to establish a unified harmonic analytical approach for all cases. To obtain a unified theory one can adopt abstract harmonic analysis. However, apart from compact groups, it is not completely clear how to relate the abstract group Fourier transform with the eigenfunction expansion. This is the challenge tackled in chapter 2. In section 2.1 we deductively introduce semidirect homogeneous vector bundles as modelling practically all cosmological geometries. Section 2.2 introduces the harmonic analytical Fourier transform on semidirect homogeneous vector bundles. While on a general homogeneous space such a transform may not exist, we show that it does exist on cosmological spacetimes and we present a fairly natural variant of it which will be used afterwards. Section 2.3 deals with the behavior of distributions on the semidirect homogeneous space under the abstract Fourier transform. We present generalizations of some classical results about finite rank distributions from $\mathbb{R}^n$ [23] to semidirect spaces. Section 2.4 provides the bridge between the harmonic analysis and the usual Fourier analysis. The most important results of chapter 2 for applications are obtained in section 2.5 where we find the general form of the invariant bi-distributions on semidirect homogeneous spaces.

If there is a single word describing the aims of the third chapter then it is the word 'explicit', referring to making explicit results of abstract harmonic analysis to the groups arising in homogeneous cosmology. In chapter 2 we have in particular established that in nearly all cases one deals with a semidirect product group $G = \Sigma \rtimes O$ where O is either of $SO(3)$, $SO(2)$, $\{1\}$, and the role of Σ is played by Bianchi I-IX groups $Bi(N)$ and their quotients $Bi(N)/\Gamma$ by discrete normal subgroups Γ. The spaces of maximal symmetry with $O = SO(3)$ are the FRW spaces, which are described by isometry groups $SO(4)$, $E(3)$ or $SO^+(1,3)$. The spaces with one rotational symmetry are described by $O = SO(2)$ and are called LRS spaces. And finally the purely homogeneous spaces are given by trivial isotropy groups $O = \{1\}$. The isometry groups of FRW spaces are classical groups and their harmonic analysis is also a classical subject. The harmonic analysis of Bianchi I-III,VIII-IX groups has been derived explicitly in the literature [56],[21],[47]. Little is known about groups Bianchi IV-VII beyond the structure of their Lie algebras which are semidirect products of Abelian algebras $\mathbb{R}^2$ and $\mathbb{R}$. Even less is known about the semidirect products of Bianchi groups with $SO(2)$ describing LRS models. We give the explicit description of the harmonic analysis of solvable Bianchi groups I-VII in a uniform manner.

The final step in establishing explicit Fourier transform theory is to obtain an explicit spectral theory for the Laplace operator Δ associated with any left invariant Riemannian metric. In this work we will give such an explicit description of the spectrum and eigenfunctions of Δ in terms of arbitrary left invariant Riemannian metric on Σ for the line bundle over Bianchi I-VII groups. The contents of the chapter goes as follows. The dual spaces of the relevant groups are constructed, i.e., the equivalence classes of unitary irreducible representations. This is done by means of the "Mackey machine". Next a look is given at the co-adjoint orbits of the groups in the sense of the Kirillov theory, and it is described explicitly how the cross sections could be chosen to parameterize the dual space. An explicit Plancherel formula is given for all relevant groups. In Section 3.6 the spectrum and the eigenfunctions of Δ acting on the line bundle are found explicitly. And it is shown that these eigenfunctions are complete. This is the point where the bridge designed in chapter 2 is constructed explicitly.

Chapter 4 presents some applications of the mode decomposition for the quantum fields. More precisely, it deals with the 2-point function ω_2 of a quasifree state of a vector valued CCR quantum field. Mode decomposition provides a parameterization of the 2-point functions in terms of mode coefficient distributions. Now several questions arise concerning how the usual properties of quasifree states like purity, homogeneity or the Hadamard property translate into the language of mode coefficients. We extend related results which have been obtained previously by [32],[52],[20],[42] for the more special FRW scenario to general homogeneous cosmologies. Section 4.2 provides the parameterization of the 2-point function of a quasifree state in terms of mode coefficients. A necessary and sufficient condition is given for the state to be pure. In Section 4.3 we apply the results on invariant bi-distributions found in chapter 2 to quasifree states of a quantum field. The general form of a homogeneous state is given in terms of modes and it is indicated how this simplifies if the spacetime is more symmetric (e.g., FRW). Finally in Section 4.4 it is shown that for a wide variety of mode solutions the positive frequency part of the propagator has the Hadamard form.

Chapter 1

Mode decomposition of hyperbolic fields

1.1 Linear Hyperbolic Fields

It is generally believed that the forces of nature are described by tensor and spinor fields. A geometrical generalization of those are the vector bundle fields, i.e., fields as smooth sections of some vector bundles. In general relativity one works mainly on a 4-dimensional Lorentzian smooth manifold (M, g) which is called a spacetime. We will be concerned with hyperbolic fields given by a wave equation, hence we put an additional constraint on the spacetime (M, g) to be globally hyperbolic, so that the Cauchy problem of the wave equation is well-posed. For simplicity only linear fields will be discussed here. For the reduction of the Maxwell and Proca fields to linear hyperbolic fields we refer the reader to [20],[2]. We summarize the basic setup of the the linear hyperbolic fields in the following section.

Let V be an n-dimensional vector space. Let $\mathcal{T} \xrightarrow{\pi} M$ be a vector bundle with standard fiber V and with a pseudo-Riemannian metric $\langle u, v\rangle_{\mathfrak{g}}$. Let further $\mathcal{E}(\mathcal{T}) = C^\infty(\mathcal{T})$ and $\mathcal{D}(\mathcal{T}) = C_0^\infty(\mathcal{T})$ be the spaces of smooth sections and of those with compact support, correspondingly. Let ∇ be a metric connection on $\mathcal{T}$ and $\Box^\nabla$ the associated d'Alambert operator on $\mathcal{E}(\mathcal{T})$. Define the field operator to be the normal hyperbolic operator $D =$

$\Box^\nabla + m^\star(x)$ acting on $\mathcal{E}(\mathcal{T})$, where $m^\star \in C^\infty(M)$ is a generalization of the usual mass term m^2 which now can also contain the coupling term ξR. Note that because differential operators are support-decreasing, $D\mathcal{D}(\mathcal{T}) \subset \mathcal{D}(\mathcal{T})$. A free linear hyperbolic field $\phi \in \mathcal{E}(\mathcal{T})$ is a solution of the field equation $D\phi = 0$.

Being a globally hyperbolic spacetime, $M = \mathcal{I} \times \Sigma$, where $\mathcal{I} \subseteq \mathbb{R}$ is an interval, and for each $t \in \mathcal{I}$ the hypersurface $\Sigma_t \sim \Sigma$ is a 3-dimensional embedded Riemannian submanifold, which is spacelike with respect to g and is a Cauchy surface in the sense described below. Thanks to [3] one can choose a smooth global time function t and a coordinate atlas such that $x = (t, \vec{x}) = (x_0, x_1, x_2, x_3)$ where $t \in \mathcal{I}$ and $\vec{x} \in \Sigma$, i.e., Σ_t are equal t hypersurfaces. The restriction of the bundle $\mathcal{T}$ to the submanifold Σ_t will be denoted by $\mathcal{T}_t = \pi^{-1}(\Sigma_t)$. The spaces of smooth sections will be $\mathcal{E}(\mathcal{T}_t) = C^\infty(\mathcal{T}_t)$ and $\mathcal{D}(\mathcal{T}_t) = C_0^\infty(\mathcal{T}_t)$. If $i_t : \mathcal{T}_t \to \mathcal{T}$ is the identical embedding, then its pullback i_t^* is the restriction map for objects on $\mathcal{T}$ to $\mathcal{T}_t$. In particular $i_t^* : \mathcal{E}(\mathcal{T}) \to \mathcal{E}(\mathcal{T}_t)$ and $i_t^* : \mathcal{D}(\mathcal{T}) \to \mathcal{D}(\mathcal{T}_t)$ are linear surjective maps. The embedding $\pi \circ i_t \circ \pi^{-1} : M \to \Sigma$ gives rise to a natural embedding $i_t : TM \to T\Sigma$ and of all tensor bundles (using the same symbols i_t, i_t^* for different restrictions in the spirit of polymorphism should not lead to a confusion). The Riemannian metric h on $T\Sigma$ will be $h = -i_t^*(g)$, with minus sign here because of the signature convention $(+,-,-,-)$. The restriction $i_t^*(\nabla) = \nabla_{i_t^*(.)} = \nabla^t$ is a metric connection on $\mathcal{T}_t$. The associated Laplace operator $\Delta_t = \Delta^{\nabla^t}$ is an elliptic operator on $\mathcal{E}(\mathcal{T}_t)$ (we choose the signature of the fiber metric such that $-\Delta_t$ is a positive operator). The restriction of the field operator D to $\mathcal{E}(\mathcal{T}_t)$ will be denoted by $D_{\Sigma_t} = -\Delta_t + m^\star(x)$.

An existence and uniqueness theorem [2],[25],[17] for wave operators tells that the Cauchy problem is well posed: there exists a bijective linear map $\mathcal{E}(\mathcal{T}_t) \oplus \mathcal{E}(\mathcal{T}_t) \ni (f_0, f_1) \to \jmath(f_0, f_1) = f \in \mathcal{E}(\mathcal{T})$ of the field equation $Df = 0$ such that $f_0 = i_t^*(f)$ and $f_1 = i_t^*(\nabla_t f)$, where $\nabla_t = \nabla_{\frac{\partial}{\partial t}}$. Furthermore, there exist unique Green's operators $E^\pm : \mathcal{D}(\mathcal{T}) \to \mathcal{E}(\mathcal{T})$ satisfying $DE^\pm = E^\pm D = id_{\mathcal{D}(\mathcal{T})}$ and $supp\{G^\pm f\} \subset J^\pm(supp\{f\})$ for all $f \in \mathcal{D}(\mathcal{T})$. Here $J^\pm(N)$ with a subset $N \subset M$ denotes the causal future/past of N. Define by $E = E^+ - E^-$ the *propagator* of D, which satisfies $DE = ED = 0$. Now $Sol(\mathcal{T}) = \jmath(\mathcal{E}(\mathcal{T}_t) \oplus \mathcal{E}(\mathcal{T})_t)$ and $Sol_0(\mathcal{T}) = \jmath(\mathcal{D}(\mathcal{T}_t) \oplus \mathcal{D}(\mathcal{T}_t))$ will denote correspondingly the spaces of all smooth solutions, and of those satisfying $supp\{f\} \cap \Sigma_t$ compact for all $t \in \mathcal{I}$, respectively. Then $E\mathcal{D}(\mathcal{T}) \subset Sol_0(\mathcal{T})$. There is a symplectic form on $Sol_0(\mathcal{T})$:

$$\sigma(u, v) = \int_{\Sigma_t} d\mu_h \left[\langle i_t^*(u), i_t^*(\nabla_t v) \rangle_g - \langle i_t^*(\nabla_t u), i_t^*(v) \rangle_g \right], \, \forall u, v \in Sol_0(\mathcal{T}), \, \forall t \in \mathcal{I},$$

where $h = -i_t^*(g)$ is the induced Riemannian metric on Σ_t. That this is conserved (analogous to a Wronskian in ODE) can be seen by considering the Green's identity for $u, v \in Sol_0(\mathcal{T})$ on the regular cylindric region $U = (t_1; t_2) \times \Sigma \subset M$ for any $t_1 \neq t_2$,

$$0 = \int_U d\mu_g \left[\langle u, Dv \rangle_g - \langle Du, v \rangle_g \right] =$$

$$= \int_{\partial U} d\mu_h \left[\langle i_t^*(u), i_t^*(\nabla_t v) \rangle_g - \langle i_t^*(\nabla_t u), i_t^*(v) \rangle_g \right], \forall u, v \in Sol_0(\mathcal{T}).$$

This identity also helps us along with Green's operators to find the explicit form of the map $\jmath$. Given any $v \in Sol_0(\mathcal{T})$, $f \in \mathcal{D}(\mathcal{T})$, we apply it two times; once for the pair $v, u = E^+(f)$ on the region $U^+ = (-\inf\{\mathcal{I}\}; t)$ and once for the pair $v, u = E^-(f)$ on the region $U^+ = (t; \inf\{\mathcal{I}\})$. Summing up the resulting two identities and using the support properties of $E^\pm$ we arrive at

$$\int_M d\mu_g \langle v, f \rangle_g = \sigma(v, E(f)). \tag{1.1}$$

We see that the functional $\sigma(v, E(.)) : \mathcal{D}(\mathcal{T}) \to \mathbb{C}$ actually is given by a smooth integral kernel, which equals v. Thus we can write symbolically

$$\jmath(f_0, f_1)[y] = \int_{\Sigma_t} d\mu_h \left[\langle f_0, \nabla_t E(y) \rangle_g - \langle f_1, E(y) \rangle_g \right], \forall f_0, f_1 \in \mathcal{D}(\mathcal{T}_t),\ y \in M,\ t \in \mathcal{I}.$$

For full details of this last computation we refer the reader to [18], where the argument is given for 1-forms, but is readily applicable to our more general case.

Proposition 1.1 *The operator $E : \mathcal{D}(\mathcal{T}) \to Sol_0(\mathcal{T})$ is surjective.*

Proof: Let $v \in Sol_0$, and let $K_v = \mathrm{supp} v \cap ([0,1] \times \Sigma)$ be the compact region of its support between times 0 and 1. Let further $\chi \in \mathcal{E}(M)$ be a smooth function which equals 1 for $t < 0$ and 0 for $t > 1$. Denote $v^- = -v\chi$ and $v^+ = v(1-\chi)$, then $v = v^+ - v^-$. Let $f_v = Dv^+$, then $\mathrm{supp} f_v \subset K_v$ is compact, hence $f_v \in \mathcal{D}(\mathcal{T})$. The equation $f_v = Dv^+$ has a unique solution with past compact support, and it is given by $v^+ = E^+ f_v$. Now $Dv^- = -Dv + Dv^+ = f_v$, and similarly $v^- = E^- f_v$. Then $v = E^+ f_v - E^- f_v = E f_v$. The arbitrariness of χ reflects the non-injectivity of E. $\square$

1.2 Spectral mode decomposition

Henceforth we will use nomenclature introduced in the appendix without special notice. Consider the operators $D : \mathcal{D}(\mathcal{T}) \to \mathcal{D}(\mathcal{T})$ and $D_\Sigma : \mathcal{D}(\mathcal{T}_t) \to \mathcal{D}(\mathcal{T}_t)$. If $m^\star(x) \in \mathbb{R}$ everywhere on M, then by the virtue of Green's identity D and D_{Σ_t} are formally self-adjoint with respect to the inner products $(,)_M$ and $(,)_{\Sigma_t}$. We will not need the self-adjointness of D in the current work. The constructions below will pertain mainly to D_{Σ_t}. The conditions on $m^\star(x)$ for D_{Σ_t} to have a self-adjoint extension can be found in [11]. We moreover require that the operator D_{Σ_t} be lower semi-bounded. In later chapters we will be mainly interested in cosmological models, where $m^\star(x) = m^\star(t)$ is a function of time only, so that no problems arise. Below we assume self-adjoint extensions for both D and D_{Σ_t}, but for D this is only symbolic and targets simply at coherent notations.

Let D and D_{Σ_t} be extended to self-adjoint operators on $L^2(\mathcal{T})$ and $L^2(\mathcal{T}_t)$ respectively. In the rigged Hilbert spaces [43],[38],[37] $\mathcal{D}(\mathcal{T}) \subset L^2(\mathcal{T}) \subset \mathcal{D}(\mathcal{T})'$ and $\mathcal{D}(\mathcal{T}_t) \subset L^2(\mathcal{T}_t) \subset \mathcal{D}(\mathcal{T}_t)'$ operators D and D_{Σ_t} possess complete systems of eigenfunctions $\{u_\rho\}$ and $\{\zeta_\lambda\}$ satisfying

$$Du_\rho = \rho u_\rho,\ u_\rho \in \mathcal{D}(\mathcal{T})',\ \rho \in \mathbb{R},$$

$$D_{\Sigma_t}\zeta_\lambda = \lambda\zeta_\lambda,\ \zeta_\lambda \in \mathcal{D}(\mathcal{T}_t)',\ \lambda \in \mathbb{R}.$$

Denote by $\mathcal{D}(\mathcal{T})'_\rho$ and $\mathcal{D}(\mathcal{T}_t)'_\lambda$ the linear spaces of eigenfunctions corresponding to ρ and λ, respectively. Furthermore, there exists an isomorphism

$$L^2(\mathcal{T}_t) = \int_{\mathbb{R}}^{\oplus} d\nu(\lambda) H(\lambda), \tag{1.2}$$

where

$$D_{\Sigma_t}|_{H(\lambda)} = \lambda,$$

and $d\nu(\lambda)$ is a positive measure. Each $H(\lambda)$ is continuously embedded in $\mathcal{D}(\mathcal{T}_t)'_\lambda$.

The eigenfunction expansion of D will be the map

$$\mathcal{D}(\mathcal{T}) \ni f \to \tilde{f}_\rho \in (\mathcal{D}(\mathcal{T})'_\rho)'^*,\ \rho \in \mathbb{R},$$

(X'^* denotes the space of continuous antilinear functionals on the space X) where $\tilde{f}_\rho$ is defined by

$$\tilde{f}_\rho(u_\rho) = \bar{u}_\rho(f),\ \forall f \in \mathcal{D}(\mathcal{T}),\ u_\rho \in \mathcal{D}(\mathcal{T})'_\rho.$$

(Here we defer a little from Gelfand's notations who puts $\tilde{f}_\rho(u_\rho) = u_\rho(f)$.)The expansion of D_{Σ_t} on $\mathcal{D}(\mathcal{T}_t)$ is constructed similarly. Note that D_{Σ_t} is an elliptic operator, hence $\mathcal{D}(\mathcal{T}_t)'_\lambda \subset \mathcal{E}(\mathcal{T}_t)$.

If each $\mathcal{D}(\mathcal{T}_t)'_\lambda$ is finite dimensional (eigenvalue λ has a finite multiplicity N_λ), then

$$H(\lambda) = \mathcal{D}(\mathcal{T}_t)'_\lambda,\ \dim H(\lambda) = N_\lambda.$$

Choose $\{\zeta_{\lambda,i}\}_{i=1}^{N_\lambda}$ be a an orthonormal basis in $\mathcal{D}(\mathcal{T}_t)'_\lambda$ (orthonormality understood in $H(\lambda)$). Then $(\mathcal{D}(\mathcal{T}_t)'_\lambda)' \sim \mathbb{C}^{N_\lambda}$ by the bijective linear map

$$\tilde{f}(\zeta_\lambda) = \tilde{f}\left(\sum_{i=1}^{N_\lambda} c_i \zeta_{\lambda,i}\right) \to \{\tilde{f}_i = \tilde{f}(\zeta_{\lambda,i})\}_{i=1}^{N_\lambda},\ \forall \tilde{f} \in (\mathcal{D}(\mathcal{T}_t)'_\lambda)',$$

where each $\tilde{f}_i \in \mathbb{C}$. In particular, if $\tilde{f}_\lambda$ is the mode expansion of $f \in \mathcal{D}(\mathcal{T}_t)$, then the map

$$\mathcal{D}(\mathcal{T}_t) \ni f \to \tilde{f}_\lambda \to \{\tilde{f}_{\lambda,i}\} \in \int_{\mathbb{R}}^{\oplus} d\nu(\lambda)\mathbb{C}^{N_\lambda} \tag{1.3}$$

will serve as a Fourier transform on $\mathcal{D}(\mathcal{T}_t)$. Define

$$Spec\{D_{\Sigma_t}\} = \text{supp}\{d\nu\},$$

and

$$\tilde{\Sigma} = \{(\lambda, i)\colon\ \lambda \in Spec\{D_{\Sigma_t}\},\ i = 1, ..., N_\lambda\}.$$

Define the spectral measure on $\tilde{\Sigma}$ as

$$d\mu(\alpha) = d\nu(\lambda) \times d\sharp(i),$$

where $d\sharp$ is the counting measure. The map (Eq.1.3) can be reformulated as

$$\tilde{f}(\alpha) = \mathcal{F}[f](\alpha),\ f \in \mathcal{D}(\mathcal{T}_t).$$

Then the formula (Eq.1.2) arises a Plancherel formula

$$(f, h)_{Stat_t} = \int_{\tilde{\Sigma}} d\mu(\alpha)\bar{\tilde{f}}(\alpha)\tilde{g}(\alpha),$$

and a Peter-Weyl (or Fourier inversion) formula

$$f(x) = \int_{\tilde{\Sigma}} d\mu(\alpha) \tilde{f}(\alpha) \zeta_\alpha(x), \tag{1.4}$$

which holds in the L^2-sense so far. In our cases of interest this convergence will be in the compact topology.

However, if $\mathcal{D}(\mathcal{T}_t)'_\lambda$ is infinite dimensional, more delicate tools are needed to obtain a Fourier transform with desired properties. Such tools naturally include an investigation of symmetries of the underlying geometrical structure, and this is the subject of the harmonic analysis. We will often use the formal structure (Eq.1.4) without mentioning a concrete realization, assuming that this is possible. For the cases of our interest we will indeed find a realization and thus complete the task.

In the theory of Fourier transform, and in particular in the Euclidean case, the Paley-Wiener theorems describe the functional analytical structure of the image $\mathcal{F}[\mathcal{D}(\mathcal{T}_t)]$ of the test function space under the action of the Fourier transform. This description is very useful when analyzing the situation in the Fourier space. Unfortunately there is no (at least known to us) general Paley-Wiener argument valid for any Fourier transform arisen in this manner, and the proofs of the existing ones are rather structure-specific. In applications we would like, however, to obtain results which are valid in a large variety of cases, and therefore we will introduce a notion of 'conventional' Fourier transform which consists of a number of assumptions pertaining to the analytical properties of a given Fourier transform. Many of our later results will be valid under the assumption that the eigenvalue expansion of the operator Δ_t gives a conventional Fourier transform.

Definition 1.1 *A Fourier transform $\mathcal{F}$ given by the eigenfunction expansion against a complete system $\{\zeta_\alpha\}_{\alpha \in \tilde{\Sigma}}$ will be called conventional if*

(i) The Fourier space (or momentum space) $\tilde{\Sigma}$ is a manifold consisting of $n = \dim V$ components, $\tilde{\Sigma} = \bigcup_{i=1}^n \tilde{\Sigma}^i$, and each component $\tilde{\Sigma}^i$ is either a discreet set or an (not necessarily connected) analytical manifold

(ii) The eigenvalue $\lambda(\alpha)$ is an analytic function on $\tilde{\Sigma}$

(iii) The range $\mathcal{F}[\mathcal{D}(\mathcal{T}_t)]$ is a subspace of the space of analytic functions $\tilde{f}(\alpha)$ on $\tilde{\Sigma}$ with

rapid decay in λ.

(iv) There is an involution $\alpha \to -\alpha$ *on* $\tilde{\Sigma}$ *such that* $\zeta_{-\alpha} = \bar{\zeta}_\alpha$.

Note that it follows $\lambda(-\alpha) = \lambda(\alpha)$. In the next chapter we will give harmonic analytical justifications for such a 'conjecture' and will show that this property holds at least for the majority of popular cosmological models.

Further in this chapter we will be mainly interested in the space of weak solutions of the field equation, $\mathcal{D}(\mathcal{T})'_0$, and will try to find a convenient characterization of it. In particular we will be looking for a complete system of solutions $\{u_\alpha\}$ spanning $\mathcal{D}(\mathcal{T})'_0$ and being in addition well handled (i.e., smooth, explicit etc.). One means of doing this is to look at a subspace of $\mathcal{D}(\mathcal{T})'_0$ which consist of solutions $f(x) = a(t)b(\vec{x})$, $a \in C^\infty(\mathcal{I})$, $b \in \mathcal{E}(\mathcal{T}_t)$. Then under fortunate circumstances the field equation breaks apart into two lower dimensional elliptical eigenproblems, which are much easier to deal with. Which are those circumstances and whether such solutions span $\mathcal{D}(\mathcal{T})'_0$, and related questions, are the matter of the problem of variable separation. In the next sections we will find out which cases this is possible and how to perform it.

1.3 Separation of variables

As discussed in the previous section, we would like to span the space $\mathcal{D}(\mathcal{T})'_0$ of weak solutions of the field equation by a family of easily computable smooth solutions $\{u_\alpha\}$. In this section we will see when and how one can perform this for the smooth solutions $Sol_0(\mathcal{T})$. The necessary requisites for this will be predominantly geometric requirements. In the next section we will show that under additional functional analytical assumptions the procedure can be extended to $\mathcal{D}(\mathcal{T})'_0$ in a natural way.

Definition 1.2 *Let* S *be a topological (complex) vector space with closure* $\bar{S} \supseteq S$, $\mathfrak{M}$ *a measure space with measure* $d\mathfrak{m}$. *An* $\mathfrak{M}$*-measurable family* $\{u_\alpha\}_{\alpha\in\mathfrak{M}}$ *of linearly independent elements* $u_\alpha \in \bar{S}$ *will be called a complete or spanning system for* S *if for any* $v \in S$ *there exists a unique (modulo null-supported functions)* $\mathfrak{M}$*-measurable function*

$a^v : \mathfrak{M} \to \mathbb{R}$ ($a^v : \mathfrak{M} \to \mathbb{C}$) such that

$$v = \int_{\mathfrak{M}} d\mathfrak{m}(\alpha) a^v(\alpha) u_\alpha,$$

with the integral converging in this topology.

Note that uniqueness is implied already by the linear independence. We will always take $\mathfrak{M}$ to be *minimal*, i.e., there exists no subset $A \subset \mathfrak{M}$ with $\mathfrak{m}(A) > 0$ such that $a^v(A) = 0$ for all $v \in S$.

$\mathcal{E}(\mathcal{T})$ is a closed topological vector space with the topology of compact convergence, and $Sol(\mathcal{T})$ and $Sol_0(\mathcal{T})$ are linear subspaces. $\mathcal{D}(\mathcal{T})'$ is a another closed topological vector space with its distributional topology, and $\mathcal{D}(\mathcal{T})'_0$ is a linear subspace. Through the natural embedding, $\mathcal{E}(\mathcal{T})$ is also a subspace of $\mathcal{D}(\mathcal{T})'$, and the compact topology of $\mathcal{E}(\mathcal{T})$ is stronger than the distributional topology of $\mathcal{D}(\mathcal{T})'$ ([15],Chapter XVII). Therefore if $\{u_\alpha\}_{\alpha\in\mathfrak{M}}$ is a complete system for $Sol(\mathcal{T})$ with compact topology, then it is such also with the weaker distributional topology.

A spanning system $\{u_\alpha\}_{\alpha\in\mathfrak{M}}$ of $Sol_0(\mathcal{T})$ of the form $u_\alpha = T_\alpha X_\alpha$, where $T_\alpha \in C^\infty(\mathcal{I})$ and $X_\alpha \in \mathcal{E}(\mathcal{T}_t)$, such that $Du_\alpha = 0$, will be called a *complete (time-)variable separated system of solutions* (or shorter, *separating system*).

We will assume that a Fourier transform $\mathcal{F}$ on $\mathcal{D}(\mathcal{T}_t)$ is specified by means of the spectral decomposition of D_{Σ_t} as described in the previous section. The system of eigenfunctions $\{\zeta^t_\alpha\}$ of Δ_t, with the Fourier space $\tilde{\Sigma}_t$ and the spectral measure $d\mu(\alpha)$ on it, provide a spanning system for $\mathcal{D}(\mathcal{T}_t)$ by means of the Fourier inversion (or Peter-Weyl) formula. Below we will come across the question of a spectral theory of formally non-self-adjoint, i.e., asymmetric differential operators of type $a(x)D_{\Sigma_t}$. As a rule, the eigenfunction problems of asymmetric (aside from unitary) operators are ill-posed, and eigenfunctions do not comprise a complete system, but there are rare exceptions. At this point we have to admit the non-exhaustiveness of our treatment, as we do not analyze this possibility. We will loosely rule out the possibility of such operators to have a well-posed eigenfunction problem.

A small remark will be useful later in the section.

Remark 1.1 *If* $\{T_\alpha X_\alpha\}_{\alpha\in\mathfrak{M}}$ *is a separating system for* $Sol_0(\mathcal{T})$ *with compact topology, then for each* $t \in \mathcal{I}$, *both the families* $\{T_\alpha(t)X_\alpha\}_{\alpha\in\mathfrak{M}}$ *and* $\{\dot{T}_\alpha(t)X_\alpha\}_{\alpha\in\mathfrak{M}}$ *are spanning systems for* $\mathcal{D}(\mathcal{T}_t)$. *In particular, for each* $\vec{x} \in \Sigma_t$, *the family* $\{X_\alpha(\vec{x})\}_{\alpha\in\mathfrak{M}}$ *contains a (possibly redundant) basis of* V.

The assertions are relatively obvious in the view of the fact, that the restriction maps $i_t^*, i_T^* \circ \nabla_t : Sol_0(\mathcal{T}) \to \mathcal{D}(\mathcal{T}_t)$ are surjective, and hence a spanning system for $Sol_0(\mathcal{T})$ must give a spanning system for the Cauchy data $\mathcal{D}(\mathcal{T}_t) \oplus \mathcal{D}(\mathcal{T}_t)$ on Σ_t.

Proposition 1.2 *Let* $\{T_\alpha X_\alpha\}$ *be a separating system for* $Sol_0(\mathcal{T})$. *Then for* $d\mathfrak{m}$*-almost each* $\alpha \in \mathfrak{M}$ *there exists at least one* $\mathfrak{M} \ni \beta \neq \alpha$ *such that* $X_\alpha = X_\beta$. *It follows that* T_α *and* T_β *are linearly independent.*

Proof: First we note that whenever $X_\alpha = X_\beta$ for $\alpha \neq \beta$ it follows that T_α and T_β are linearly independent, otherwise the linear independence of $T_\alpha X_\alpha$ and $T_\beta X_\beta$ would be violated. This proves the second statement. For any $u \in Sol_0(\mathcal{T})$ we write

$$u = \int_{\mathfrak{M}} d\mathfrak{m}(\alpha) a^u(\alpha) T_\alpha X_\alpha.$$

The Cauchy data (u_0, u_1) of u on Σ_t are given by

$$u_0 = \int_{\mathfrak{M}} d\mathfrak{m}(\alpha) a^u(\alpha) T_\alpha(t) X_\alpha,$$

$$u_1 = \int_{\mathfrak{M}} d\mathfrak{m}(\alpha) a^u(\alpha) \dot{T}_\alpha(t) X_\alpha.$$

For a pair $\eta = (\eta_0, \eta_1) \in \mathbb{R}^2$ ($\mathbb{C}^2$) set $(v_0^\eta, v_1^\eta) = (\eta_0 u_0, \eta_1 u_1)$. Then $v^\eta = \jmath(v_0^\eta, v_1^\eta)$ will be a solution $v^\eta \in Sol_0(\mathcal{T})$.

Let $\mathfrak{M}_1, \mathfrak{M}_2 \subset \mathfrak{M}$ be the subsets for which $X_\alpha = X_\beta$ happens for no and for one or more pairs α, β, respectively. Because $\{T_\alpha X_\alpha\}$ is measurable, the sets $\mathfrak{M}_1, \mathfrak{M}_2 \subset \mathfrak{M}$ are measurable, and $\mathfrak{M}_1 + \mathfrak{M}_2 = \mathfrak{M}$. Fix a time $t \in \mathcal{I}$. By **Remark 1.1** the fields $T_\alpha(t)$ and $\dot{T}_\alpha(t)$ vanish $d\mathfrak{m}$-almost nowhere on $\mathfrak{M}$. For $\alpha \in \mathfrak{M}_1$, $\alpha \neq \beta \in \mathfrak{M}$ we have $T_\alpha(t)X_\alpha$ and $T_\beta(t)X_\beta$ are linearly independent in $\mathcal{E}(\mathcal{T}_t)$. For any $\alpha \in \mathfrak{M}_2$ there is at least one $\beta \in \mathfrak{M}$ such that $\beta \neq \alpha$ and $T_\alpha(t)X_\alpha$ and $T_\beta(t)X_\beta$ are linearly dependent, and all such β lie in

$\mathfrak{M}_2$. It follows in particular, that whenever

$$\int_{\mathfrak{M}} d\mathfrak{m}(\alpha) a^u(\alpha) T_\alpha(t) X_\alpha = 0$$

then

$$\int_{\mathfrak{M}_1} d\mathfrak{m}(\alpha) a^u(\alpha) T_\alpha(t) X_\alpha = \int_{\mathfrak{M}_2} d\mathfrak{m}(\alpha) a^u(\alpha) T_\alpha(t) X_\alpha = 0.$$

The same is true also with $\dot{T}_\alpha(t)$ instead of $T_\alpha(t)$

Let $u \in Sol_0(\mathcal{T})$ be a nonzero solution, and choose time t such that $u_1 \neq 0$ on Σ_t. Such a choice is always possible because static functions do not solve the field equation. Set $\eta = (0,1)$, then v^η will be a nonzero solution with $v_0^\eta = 0$ on Σ_t. Then

$$0 = \int_{\mathfrak{M}} d\mathfrak{m}(\alpha) a^{v^\eta}(\alpha) T_\alpha(t) X_\alpha = \int_{\mathfrak{M}_1} d\mathfrak{m}(\alpha) a^{v^\eta}(\alpha) T_\alpha(t) X_\alpha,$$

and because $T_\alpha(t) X_\alpha$ are linearly independent for $\alpha \in \mathfrak{M}_1$, it follows $a^{v^\eta}(\alpha) = 0$ on $\mathfrak{M}_1$. But on the other hand

$$0 = \int_{\mathfrak{M}} d\mathfrak{m}(\alpha) \left[a^{v^\eta}(\alpha) - a^u(\alpha)\right] \dot{T}_\alpha(t) X_\alpha = \int_{\mathfrak{M}_1} d\mathfrak{m}(\alpha) \left[a^{v^\eta}(\alpha) - a^u(\alpha)\right] \dot{T}_\alpha(t) X_\alpha,$$

from where it follows $a^{v^\eta}(\alpha) - a^u(\alpha) = 0$ and hence $a^u(\alpha) = 0$ $d\mathfrak{m}$-almost everywhere on $\mathfrak{M}_1$. But u was arbitrary, and because $\mathfrak{M}$ is minimal it follows $\mathfrak{m}(\mathfrak{M}_1) = 0$. $\square$

Remark 1.2 *Let two equations* $\ddot{T}(t) + F(t)\dot{T}(t) + G(t)T(t) = 0$ *and* $\ddot{T}(t) + H(t)\dot{T}(t) + J(t)T(t) = 0$ *have two common linearly independent solutions* $T(t)$ *and* $S(t)$*. Then by Liouville formula the Wronski determinant* $\det W[T,S](t)$ *evolves by*

$$\det W[T,S](t) = \det W[T,S](0) e^{-\int_0^t d\tau F(\tau)} = \det W[T,S](0) e^{-\int_0^t d\tau H(\tau)},$$

hence $F = H$ *and thereby also* $G = J$*.*

Proposition 1.3 *The solution space* $Sol_0(\mathcal{T})$ *admits a separating system if and only if there exists a covering of* $\mathcal{T}$ *by local trivializations such that the following local conditions are satisfied everywhere (metric* g *is time-separated):*

(i) $g_{00} = g_{00}(t)$*, the metric component* g_{00} *depends only on time*

(ii) the expression $\sum_{i,j=1}^{3} g^{ij}(x)\frac{\partial g_{ij}}{\partial t}(x)$ *is a function of time only*

(iii) the connection 1-form Γ *and Christoffel symbols* $\mathbf{\Gamma}_{ij}^{k}$ *satisfy*

$$\sum_{i=1}^{3} g^{ij}[\Gamma_0, \Gamma_i] = 0, \; \forall j > 0,$$

$$\sum_{i,j=1}^{3} g^{ij}\left[\Gamma_0, \frac{\partial \Gamma_j}{\partial x^i} + \Gamma_i\Gamma_j - \sum_{k=0}^{3} \mathbf{\Gamma}_{ij}^{k}\Gamma_k\right] = 0,$$

$$\Gamma_0 = \Gamma_0(t) \text{ is a function of time only}$$

(iv) the eigenfunction problem of D_{Σ_t} *on different* Σ_t *can be adjusted, so that all* $\tilde{\Sigma}_t$ *are isomorphic and the eigenfunctions* $\zeta_\alpha^t = \zeta_\alpha$ *are time-independent.*

Proof: Throughout the section we will work exclusively locally, i.e., in a local trivialization $\pi^{-1}(U) \xrightarrow{\Psi} U \times V$, $U \subset M$. Thus we identify the sections in a bundle having a typical fiber $\mathfrak{F}$ with functions in $C^\infty(U; \mathfrak{F})$. We will not keep the flag U in this section but will always understand objects as restricted to U.

The d'Alambert operator $\Box^\nabla$ on $\mathcal{E}(\mathcal{T})$ has the following local expression in terms of the connection form coefficients Γ_i and Christoffel symbols $\mathbf{\Gamma}_{ij}^{k}$,

$$\Box^\nabla = \sum_{i,j=0}^{3} g^{ij}\left[\frac{\partial^2}{\partial x^i \partial x^j} + 2\Gamma_i\frac{\partial}{\partial x^j} - \sum_{k=0}^{3}\mathbf{\Gamma}_{ij}^{k}\frac{\partial}{\partial x^k} + \frac{\partial \Gamma_i}{\partial x^j} + \Gamma_i\Gamma_j - \sum_{k=0}^{3}\mathbf{\Gamma}_{ij}^{k}\Gamma_k\right], \tag{1.5}$$

and the field operator D locally looks like

$$D = \sum_{i,j=0}^{3} g^{ij}\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=0}^{3} A^i\frac{\partial}{\partial x^i} + B + m^\star,$$

where $A_i, B \in C^\infty(U, End(V))$. To achieve a time separation we need to choose a coordinate atlas such that everywhere $g^{0i} = 0$ for $i > 0$. Then the operator D locally breaks apart into two differential operators, $D = D_t + D_{\Sigma_t}$, where

$$D_t = g^{00}\frac{\partial^2}{\partial t^2} + A^0\frac{\partial}{\partial t} + B^0,$$

and

$$D_{\Sigma_t} = \sum_{i,j=1}^{3} g^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} + \sum_{i=1}^{3} A^i \frac{\partial}{\partial x^i} + B^3 + m^\star = -\Delta_t + m^\star$$

is the restricted field operator defined earlier. $B^0, B^3 \in C^\infty(U, End(V))$ are to be seen explicitly from (Eq.1.5).

$\Rightarrow$*Necessity:* Let $\{T_\alpha X_\alpha\}$ be the separating system system. Then

$$DT_\alpha(t)X_\alpha(\vec{x}) = (D_t + D_{\Sigma_t})T_\alpha(t)X_\alpha(\vec{x}) = \ddot{T}_\alpha(t)g^{00}(x)X_\alpha(\vec{x}) + \\ +\dot{T}_\alpha(t)A^0(x)X_\alpha(\vec{x}) + T_\alpha(t)\left[B^0(x) + D_{\Sigma_t}\right]X_\alpha(\vec{x}) = 0. \tag{1.6}$$

That the metric signature is definite it follows that $g^{00}(x)$ never vanishes. We find a family of second order linear homogeneous differential equations

$$\ddot{T}_\alpha(t)g^{00}(x)X^i_\alpha(\vec{x}) + \dot{T}_\alpha(t)\left(A^0(x)X_\alpha(\vec{x})\right)^i + T_\alpha(t)\left(\left[B^0(x) + D_{\Sigma_t}\right]X_\alpha(\vec{x})\right)^i = 0$$

parameterized by the spatial coordinates $\vec{x} \in \Sigma$ and fiber indices $i = 1, ..., n$. By **Proposition 1.2** we know that there exists a $\beta \neq \alpha$ with $X_\beta = X_\alpha$ and T_β, T_α linearly independent. This means that all these equations share at least two linearly independent solutions T_α and T_β. If for some $\vec{x}$ and i, $X^i_\alpha(\vec{x}) = 0$, then the existence of two linearly independent solutions for the resulting first order equation means that

$$\left(A^0(x)X_\alpha(\vec{x})\right)^i = \left(\left[B^0(x) + D_{\Sigma_t}\right]X_\alpha(\vec{x})\right)^i = 0.$$

Otherwise, by **Remark 1.2** we find that there exist functions $F_\alpha, G_\alpha \in C^\infty(\mathcal{I})$ such that

$$\left(A^0(x)X_\alpha(\vec{x})\right)^i = g^{00}(x)F_\alpha(t)X^i_\alpha(\vec{x}),\ \left(\left[B^0(x) + D_{\Sigma_t}\right]X_\alpha(\vec{x})\right)^i = g^{00}(x)G_\alpha(t)X^i_\alpha(\vec{x}).$$

In both cases we establish that

$$g_{00}(x)A^0(x)X_\alpha(\vec{x}) = F_\alpha(t)X_\alpha(\vec{x}) \tag{1.7}$$

and

$$g_{00}(x)\left[B^0(x) + D_{\Sigma_t}\right]X_\alpha(\vec{x}) = G_\alpha(t)X_\alpha(\vec{x}). \tag{1.8}$$

Thus for each $t \in \mathcal{I}$, X_α-s must be nothing else but the joint eigenfunctions of the operators $g_{00}(x)A^0(x)$ and $g_{00}(x)\left[B^0(x) + D_{\Sigma_t}\right]$ corresponding to eigenvalues $F_\alpha(t)$ and $G_\alpha(t)$, respectively. The operator $g_{00}(x)A^0(x)$ is simply a matrix, and at each point

$x \in M$ has at most n independent eigenvectors. By **Remark 1.1**, $X_\alpha(\vec{x})$-s span V, and thereby $\{X_\alpha\}_{\alpha\in\mathfrak{M}}$ contains bases of all eigenspaces of $g_{00}(x)A^0(x)$. From (Eq.1.5) we find

$$g_{00}A^0 = 2\Gamma_0 - g_{00}\sum_{i,j=0}^{3} g^{ij}\mathbf{\Gamma}^0_{ij}, \tag{1.9}$$

and

$$g_{00}B^0 = \frac{\partial}{\partial t}\Gamma_0 + \Gamma_0^2 - \sum_{k=1}^{3}\mathbf{\Gamma}^k_{00}\Gamma_k - g_{00}\sum_{i,j=0}^{3} g^{ij}\mathbf{\Gamma}^0_{ij}\Gamma_0.$$

Now turn to the eigenfunction problem (Eq.1.8). As discussed above, for this problem to be well-posed it is necessary that the differential operator $g_{00}(x)\left[B^0(x) + D_{\Sigma_t}\right]$ is at least formally self-adjoint. But this is possible only if $g_{00}(x) = g_{00}(t)$, thus we have obtained the condition (i). Let us switch to an atlas, where the time function t is redefined such that $g_{00}(t) = 1$ (this step is not crucial, but only for convenience). It follows, that

$$\mathbf{\Gamma}^k_{00} = 0,\ \forall k > 0,$$

so we obtain

$$A^0 = 2\Gamma_0 - \sum_{i,j=1}^{3} g^{ij}\mathbf{\Gamma}^0_{ij}, \tag{1.10}$$

$$B^0 = \frac{\partial}{\partial t}\Gamma_0 + \Gamma_0^2 - \sum_{i,j=1}^{3} g^{ij}\mathbf{\Gamma}^0_{ij}\Gamma_0. \tag{1.11}$$

Combining (Eq.1.7) and (Eq.1.10) we see that $\{X_\alpha\}$-s are the eigenvectors of Γ_0, and these eigenvectors are independent of t. Hence they are also the eigenvectors of $\frac{\partial}{\partial t}\Gamma_0$, and thus by (Eq.1.11) A^0 and B^0 are simultaneously triangularizable,

$$B^0 X_\alpha(\vec{x}) = H_\alpha(x) X_\alpha(\vec{x}),$$

for some $H_\alpha \in C^\infty(M)$. We note that

$$\mathbf{\Gamma}^0_{ij} = -\frac{1}{2}\frac{\partial g_{ij}}{\partial t},$$

and denote

$$P(x) = -\sum_{i,j=1}^{3} g^{ij}(x)\mathbf{\Gamma}^0_{ij}(x) = \frac{1}{2}\sum_{i,j=1}^{3} g^{ij}(x)\frac{\partial g_{ij}}{\partial t}(x).$$

Now (Eq.1.7) and (Eq.1.8) tell us, that for each $t \in \mathcal{I}$ the operators A^0 and $D_{\Sigma_t} + B^0$

have a common system of eigenfunctions spanning $\mathcal{D}(\mathcal{T}_t)$, and therefore must commute,

$$\left[A^0, D_{\Sigma_t} + B^0\right] u = \left[A^0, D_{\Sigma_t}\right] u = 0, \, \forall u \in \mathcal{D}(\mathcal{T}_t).$$

This requires

$$A^0(x) = 2\Gamma_0(x) + P(x) = A^0(t),$$

and

$$\sum_{i=1}^{3} g^{ij}[\Gamma_0, \Gamma_i] = 0, \, \forall j > 0,$$

$$\sum_{i,j=1}^{3} g^{ij} \left[\Gamma_0, \frac{\partial \Gamma_j}{\partial x^i} + \Gamma_i \Gamma_j - \sum_{k=0}^{3} \mathbf{\Gamma}_{ij}^k \Gamma_k\right] = 0,$$

exactly as the statement. Similarly, that operators B^0 and $B^0 + D_{\Sigma_t}$ have the same eigenfunctions implies, that $[B^0, D_{\Sigma_t}] = 0$, which on its turn requires $B^0(x) = B^0(t)$, and thereby $P(x) = P(t)$ and $\Gamma_0(x) = \Gamma_0(t)$. Thus we have proven parts (ii) and (iii) of the statement. It follows further, that $H_\alpha(x) = H_\alpha(t)$, and thus the eigenfunction problem (Eq.1.8) becomes

$$D_{\Sigma_t} X_\alpha(\vec{x}) = (G_\alpha(t) - H_\alpha(t)) X_\alpha(\vec{x}).$$

This is exactly the eigenfunction problem of D_{Σ_t}, whence we conclude, that necessarily

$$\{X_\alpha\}_{\alpha \in \mathfrak{M}} \subset \{\zeta_\lambda^t\}_{\lambda \in \mathbb{R}}.$$

Therefore

$$G_\alpha(t) = H_\alpha(t) + \lambda_\alpha(t),$$

where

$$\lambda_\alpha(t) = \{\lambda \in \mathbb{R} \colon \, X_\alpha \in \mathcal{D}(\mathcal{T}_t)'_\lambda\}.$$

Now (Eq.1.6) becomes

$$\ddot{T}_\alpha(t) + F_\alpha(t) \dot{T}_\alpha(t) + G_\alpha(t) T_\alpha(t) = 0, \tag{1.12}$$

which is the mode equation for the mode T_α. We have two spanning systems for $\mathcal{D}(\mathcal{T}_t)$: $\{X_\alpha\}_{\alpha \in \mathfrak{M}}$ and $\{\zeta_\alpha^t\}_{\alpha \in \tilde{\Sigma}_t}$, and hence in each eigenspace $\mathcal{D}(\mathcal{T}_t)'_\lambda$ we can choose a basis from $\{X_\alpha\}_{\alpha \in \mathfrak{M}}$. Thus a complete eigenfunction system can be chosen among $\{X_\alpha\}_{\alpha \in \mathfrak{M}}$, proving the (iv) statement of the proposition. We are complete with the necessity.

$\Leftarrow$*Sufficiency:* Suppose all the points of the statement are satisfied. Then, as we have seen above, by (iii) A^0 and B^0 are functions of t having the same eigenvectors, and moreover, commute with Δ_t. It follows that the actions of A^0 and B^0 preserve $\mathcal{D}(\mathcal{T}_t)'_\lambda$, and thus by a Gramm-Schmidt operation the representatives ζ_α can be chosen such that they are eigenfunctions of A^0 and B^0. Thus each $\tilde{\Sigma}_\lambda$, and thereby the entire $\tilde{\Sigma}$, decomposes into n components corresponding to the eigendirections of A^0,

$$\tilde{\Sigma} = \bigcup_{i=1}^{n} \tilde{\Sigma}^i.$$

For spatially homogeneous spacetimes discussed in later sections we will give a more conceptual justification of such a subdivision in terms of the representation theory.

Let for each $\alpha \in \tilde{\Sigma}$ chose a mode solution T_α of (Eq.1.12) arbitrarily (strictly speaking, not completely arbitrarily, but such that T_α and $\bar{T}_\alpha$ are linearly independent) and consider the union of two systems

$$\{u, v\}_{\alpha\in\tilde{\Sigma}} \doteq \{u_\alpha\}_{\alpha\in\tilde{\Sigma}} \cup \{v_\alpha\}_{\alpha\in\tilde{\Sigma}},\ u_\alpha = T_\alpha\zeta_\alpha,\ v_\alpha = \bar{T}_\alpha\zeta_\alpha.$$

Choose any $\phi \in Sol_0(\mathcal{T})$. Then for each $t \in \mathcal{I}$ the restriction $i_t^*(\phi)[\vec{x}] = \phi(t, \vec{x}) \in \mathcal{D}(\mathcal{T}_t)$ can be Fourier expanded as

$$\phi(t, \vec{x}) = \int_{\tilde{\Sigma}} d\mu(\alpha)\hat{\phi}(\alpha; t)\zeta_\alpha(\vec{x}) \tag{1.13}$$

with the integral converging in $L^2(\tilde{\Sigma}, \mu)$. Hence we can differentiate under the integral,

$$D\phi(t, \vec{x}) = \int_{\tilde{\Sigma}} d\mu(\alpha) D\left[\hat{\phi}(\alpha; t)\zeta_\alpha(\vec{x})\right] = \int_{\tilde{\Sigma}} d\mu(\alpha)\left[\ddot{\hat{\phi}}(\alpha; t) + \right.$$

$$\left. + F_\alpha(t)\dot{\hat{\phi}}(\alpha; t) + G_\alpha(t)\hat{\phi}(\alpha; t)\right]\zeta_\alpha(\vec{x}) = 0,$$

where for convenience we again reparameterized t to get $g_{00} = 1$. Thus $\hat{\phi}(\alpha; t)$ is a solution of the mode equation. All solutions of the ordinary second order equation (Eq.1.12) are smooth and comprise a two complex dimensional space,

$$\hat{\phi}(\alpha; t) = a_\alpha^\phi T_\alpha(t) + b_\alpha^\phi \bar{T}_\alpha(t),\ a_\alpha^\phi, b_\alpha^\phi \in \mathbb{C}.$$

Inserting this into (Eq.1.13) we finally arrive at

$$\phi(t, \vec{x}) = \int_{\tilde{\Sigma}} d\mu(\alpha)\left[a_\alpha^\phi T_\alpha(t)\zeta_\alpha(\vec{x}) + b_\alpha^\phi \bar{T}_\alpha(t)\zeta_\alpha(\vec{x})\right],$$

which exactly means, that $\{u,v\}_{\alpha\in\tilde{\Sigma}}$ is a separating system for $Sol_0(\mathcal{T})$. (For compatibility with the definition one can concatenate u_α and v_α to a single function on the disjoint union $\tilde{\Sigma}\sqcup\tilde{\Sigma}$.) $\square$

The assertion of this proposition can be interpreted as follows. If a mode decomposition in a reasonable fashion exists for $Sol_0(\mathcal{T})$ then it is basically the mode decomposition given by the time dependent Fourier transform which we will define a few paragraphs later.

As a supplement to the proposition we make a few remarks. Let $\hat{g}_{ij} = g(\partial_i,\partial_j)$ and $\hat{h}_{ij} = h(\partial_i,\partial_j)$ be the matrices of the metrics g and h, correspondingly, in a local chart, and $\sigma_k(\hat{h})$ the eigenvalues of the symmetric matrix $\hat{h}$.

Remark 1.3 *The condition (ii) of* **Proposition 1.3** *is equivalent to*

$$\det\hat{g}(x) = -g_{00}(t)\det\hat{h}(x) = -g_{00}(t)\sigma_1(\hat{h})\sigma_2(\hat{h})\sigma_3(\hat{h}) = -g_{00}(t)e^{2\int_0^t dt' P(t')}\det\hat{h}_0(\vec{x}),$$

where $\det\hat{h}_0(\vec{x})\in C^\infty(\Sigma)$ *is a positive smooth function (the notation will become clear later).*

The assertion follows from the combination of condition (ii) with the Laplace's formula,

$$\frac{\partial}{\partial t}\det\hat{g} = \det\hat{g}\cdot Tr[\hat{g}^{-1}\frac{\partial}{\partial t}\hat{g}].$$

Because ∇ is a metric connection, the restrictions of the previous proposition imply restrictions on the fiber metric $\langle,\rangle_{\mathfrak{g}}$. In case of a tensor bundle of rank (m,n) with Levi-Civita connection, coefficients Γ_i are expressed in Christoffel symbols and the fiber metric is induced by the spacetime metric, thus the restrictions fall onto the spacetime (M,g).

Corollary 1.1 *Let a local moving frame be chosen, such that the metric* $\langle,\rangle_{\mathfrak{g}}$ *is represented by the matrix* $\hat{\mathfrak{g}}$*. Conditions (iii) of* **Proposition 1.3** *imply the following restrictions on* $\hat{\mathfrak{g}}$*:*

$$\hat{\mathfrak{g}}(x) = \hat{\mathfrak{B}}^T(t)\hat{\mathfrak{g}}^0(\vec{x})\hat{\mathfrak{B}}(t),$$

where $\hat{\mathfrak{g}}^0$ *and* $\hat{\mathfrak{B}}$ *are matrix valued smooth functions. In particular, for the a tensor bundle of rank* (m,n) *to allow for seprataion it is necessary that the spacetime metric be* *represented by a matrix*

$$\hat{g} = 1 \oplus \left(-\hat{h}_0(\vec{x})\hat{B}(t)\right),$$

where $\hat{h}_0$ and $\hat{B}$ are matrix valued smooth functions.

Proof: Locally the conservation of the metric $\nabla\langle,\rangle_{\mathfrak{g}} = 0$ can be written as

$$\frac{\partial}{\partial x^i}\hat{\mathfrak{g}} - \Gamma_i^T\hat{\mathfrak{g}} - \hat{\mathfrak{g}}\Gamma_i = 0,$$

where Γ_i are the matrices of the connection coefficients in the chosen frame. In particular, for $i = 0$ we have

$$\frac{\partial}{\partial t}\hat{\mathfrak{g}}(x) - \Gamma_0^T(t)\hat{\mathfrak{g}}(x) - \hat{\mathfrak{g}}(x)\Gamma_0(t) = 0,$$

where $\Gamma_0 = \Gamma_0(t)$ was used. The solutions of this equation are of the form

$$\hat{\mathfrak{g}}(x) = \hat{\mathfrak{B}}^T(t)\hat{\mathfrak{g}}^0(\vec{x})\hat{\mathfrak{B}}(t),$$

where

$$\hat{\mathfrak{B}}(t) = e^{\int_0^t dt'\Gamma_0(t')}, \tag{1.14}$$

and $\hat{\mathfrak{g}}^0(\vec{x})$ is a smooth symmetric matrix field on Σ.

Now if we identify the tensor space $(T_pM)_n^m$ with an 4^{n+m} dimensional vector space V using a suitable bases, then each matrix Γ_i will be a $4^{n+m-1} \times 4^{n+m-1}$ matrix of blocks, with blocks being the Christoffel symbols $\mathbf{\Gamma}_i$ for contravariant indices and $-\mathbf{\Gamma}_i^T$ for covariant indices. $\Gamma_0 = \Gamma_0(t)$ means $\mathbf{\Gamma}_0 = \mathbf{\Gamma}_0(t)$. With our time-separated metric we have

$$\hat{g} = 1 \oplus -\hat{h}.$$

One can find

$$\mathbf{\Gamma}_0 = 0 \oplus \left(\frac{1}{2}\hat{h}^{-1}\frac{\partial \hat{h}}{\partial t}\right) = \mathbf{\Gamma}_0(t) = 0 \oplus \hat{A}(t)$$

for some smooth 3x3 matrix $\hat{A}(t)$. The solution is

$$\hat{h}(x) = \hat{h}_0(\vec{x})e^{2\int_0^t dt'\hat{A}(t')} = \hat{h}_0(\vec{x})\hat{B}(t),$$

for smooth symmetric commuting matrix fields $\hat{h}_0(\vec{x})$ and $\hat{B}(t)$. $\square$

Now the notation $\det \hat{h}_0$ of **Remark 1.3** becomes clear, and we see that

$$\det \hat{B}(t) = e^{2\int_0^t dt' P(t')}$$

for a tensor bundle. Note that for the scalar field conditions (iii) are trivially satisfied and do not restrict the spacetime.

Remark 1.4 *For the volume form measure $d\mu_h$ on Σ_t we have locally*

$$d\mu_h(\vec{x}) = \sqrt{\det \hat{h}(t,\vec{x})} dx^1 dx^2 dx^3.$$

By **Remark 1.3** *we have*

$$\det \hat{h}(t,\vec{x}) = e^{2\int_0^t dt' P(t')} \det \hat{h}_0(\vec{x}),$$

hence

$$d\mu_h(\vec{x}) = e^{\int_0^t dt' P(t')} d\mu_{h_0}(\vec{x}),$$

where

$$d\mu_{h_0}(\vec{x}) = \sqrt{\det \hat{h_0}(\vec{x})} dx^1 dx^2 dx^3.$$

Henceforth by stating that a mode decomposition of $Sol_0(\mathcal{T})$ exists we will mean that the assumptions of the **Proposition 1.3** are satisfied and the corresponding covering is chosen. We are ready to formulate precisely the time dependent Fourier transform. Note that although ζ_α are t-independent, the spatial metric h and the fiber metric $\langle , \rangle_{\mathfrak{g}}$ depend on t, and ζ_α are not orthonormal with respect to the measure $d\mu_h$ for all t simultaneously. At this point we appoint once and forever to normalize ζ_α such that they are orthonormal at $t = 0$. Or equivalently, they are orthonormal with respect to the measure $d\mu_{h_0}$ of **Remark 1.4** and the fiber metric $\mathfrak{g}^0$ of **Corollary 1.1**.

Definition 1.3 *For $f \in \mathcal{D}(\mathcal{T})$ we define the time dependent Fourier transform $\tilde{f}(t,\alpha) = \mathcal{F}[f(t,.)](\alpha) \in C_0^\infty\left(\mathcal{I}, \tilde{\mathcal{D}}(\tilde{\Sigma})\right)$ by*

$$\mathcal{F}[f(t,.)](\alpha) = \int_{\Sigma_t} d\mu_{h_0}(\vec{x}) \langle \bar{\zeta}_\alpha(\vec{x}), f(t,\vec{x})\rangle_{\mathfrak{g}^0}.$$

Here we note another important corollary, which will be useful later. It will give the time dependent Plancherel formula.

Corollary 1.2 *Suppose the assumptions of* **Proposition 1.3** *are satisfied, and the corresponding covering is chosen. Then for all* $f \in \mathcal{D}(\mathcal{T})$

$$(\zeta_\alpha, f(t,.))_{\Sigma_t} = I_\alpha(t)\mathcal{F}[h(t,.)](\alpha)$$

and the time dependent Plancherel formula for the time-dependent Fourier transform is given by

$$(f(t,.), h(t,.))_{\Sigma_t} = \int_{\tilde{\Sigma}} d\mu(\alpha) I_\alpha(t)\overline{\mathcal{F}[f(t,.)]}(\alpha)\mathcal{F}[h(t,.)](\alpha),$$

where

$$I_\alpha(t) = e^{\int_0^t dt' F_\alpha(t')}.$$

Proof: By **Remark 1.4**

$$(f(t,.), h(t,.))_{\Sigma_t} = \int_{\Sigma_t} d\mu_h(\vec{x})(f(t,.), h(t,.))_{\mathfrak{g}} = e^{\int_0^t dt' P(t')} \int_{\Sigma_t} d\mu_{h_0}(\vec{x})(f(t,.), h(t,.))_{\mathfrak{g}}.$$

At the same time by **Corollary 1.1** we have

$$(f(t,.), h(t,.))_{\mathfrak{g}} = (\hat{\mathfrak{B}}(t)f(t,.), \hat{\mathfrak{B}}(t)h(t,.))_{\mathfrak{g}^0}. \tag{1.15}$$

Because we have normalized ζ_α with respect to $d\mu_{h_0}$ and $\mathfrak{g}^0$, the conventional Plancherel formula holds for them,

$$\int_{\Sigma_t} d\mu_{h_0}(\vec{x})(f(t,.), h(t,.))_{\mathfrak{g}^0} = \int_{\tilde{\Sigma}} d\mu(\alpha)\overline{\mathcal{F}[f(t,.)](\alpha)}\mathcal{F}[h(t,.)](\alpha).$$

Combining these three formulas we find

$$(f(t,.), h(t,.))_{\Sigma_t} = e^{\int_0^t dt' P(t')} \int_{\tilde{\Sigma}} d\mu(\alpha)\overline{\mathcal{F}[\hat{\mathfrak{B}}(t)f(t,.)](\alpha)}\mathcal{F}[\hat{\mathfrak{B}}(t)f(t,.)](\alpha).$$

Meanwhile

$$(\zeta_\alpha, f(t,.))_{\Sigma_t} = e^{\int_0^t dt' P(t')} \int_{\Sigma_t} d\mu_{h_0}(\vec{x})(\hat{\mathfrak{B}}(t)\zeta_\alpha, \hat{\mathfrak{B}}(t)h(t,.))_{\mathfrak{g}^0}.$$

By definition

$$\hat{\mathfrak{B}}(t)\zeta_\alpha = e^{\int_0^t dt'\Gamma_0(t')}\zeta_\alpha = e^{\frac{1}{2}\int_0^t dt'[A^0(t')-P(t')]}\zeta_\alpha = e^{\frac{1}{2}\int_0^t dt'[F_\alpha(t')-P(t')]}\zeta_\alpha,$$

whence

$$(\zeta_\alpha, f(t,.))_{\Sigma_t} = e^{\frac{1}{2}\int_0^t dt'[F_\alpha(t')+P(t')]}\mathcal{F}[\hat{\mathfrak{B}}(t)f(t,.)](\alpha).$$

Finally

$$\hat{\mathfrak{B}}(t)f(t,\vec{x}) = \int_{\tilde{\Sigma}} d\mu(\alpha)\mathcal{F}[f(t,.)](\alpha)\hat{\mathfrak{B}}(t)\zeta_\alpha(\vec{x}) = \int_{\tilde{\Sigma}} d\mu(\alpha)\mathcal{F}[f(t,.)](\alpha)e^{\frac{1}{2}\int_0^t dt'[F_\alpha(t')-P(t')]}\zeta_\alpha,$$

thus

$$\mathcal{F}[\hat{\mathfrak{B}}(t)f(t,.)](\alpha) = e^{\frac{1}{2}\int_0^t dt'[F_\alpha(t')-P(t')]}\mathcal{F}[f(t,.)](\alpha).$$

The assertions now easily follow. □

At last we compute the spectra of operators A^0 and B^0 for the tensor bundle to find the functions F_α and H_α. In view of **Corollary 1.1** the function $P(t)$ becomes

$$P(t) = \frac{1}{2}Tr\left[\hat{B}^{-1}(t)\frac{\partial \hat{B}}{\partial t}(t)\right].$$

Then

$$Spec\{A^0\} = 2Spec\{\Gamma_0\} + P(t),$$

$$Spec\{B^0\} = \left\{\dot{\sigma}(t) + \sigma^2(t) + \sigma(t)P(t) \colon \ \sigma \in Spec\{\Gamma_0\}\right\}.$$

As a useful example we calculate these spectra for the scalar and 1-form fields on uniformly expanding (e.g., FRW) manifolds,

$$ds^2 = dt^2 - a^2(t)d\sigma^2(\vec{x}).$$

Here the matrix $\hat{B}(t) = a^2(t)1$, and hence

$$\hat{A}(t) = \frac{\partial}{\partial t}\ln a(t)1 = H(t)1,$$

and

$$P(t) = 3H(t), \ H(t) = \frac{\dot{a}(t)}{a(t)}.$$

For scalar case $n = m = 0$ and we have

$$Spec\{\Gamma_0\} = \{0\},$$

thus

$$Spec\{A^0\} = \{3H(t)\},\ Spec\{B^0\} = \{0\},$$

as well known. For the 1-form case, $m = 0$, $n = 1$, we have

$$Spec\{\Gamma_0\} = \{0, -H(t)\},$$

and thereby

$$Spec\{A^0\} = \{3H(t), H(t)\},\ Spec\{B^0\} = \{0, -\dot{H}(t) - 2H^2(t)\},$$

where the first members are similar to the scalar case and represent the scalar modes, but second ones represent the transversal and longitudinal modes.

As we have seen, for the separation it is necessary that the evolution of the metric be represented by linear transformations. If the connection also satisfies such a condition in a suitable sense, than the operator D_{Σ_t} is essentially the same at every t up to some scale factors. (Maybe the condition (iii) of the main proposition already implies such a restriction on the connection, but we are not sure yet.) This will be the case for all our bundles of interest, and it will provide analytical advantages. To summarize what we expect precisely we give the following definitions.

Definition 1.4 *We will say that the operator D_{Σ_t} has a* **strictly uniform** *spectrum over time if there exists a lower semi-bounded function $\omega(\alpha)$ on $\tilde{\Sigma}$, a positive smooth function $C(t) > 0$ and a smooth function $\tilde{m}^\star(t)$ such that $\lambda_\alpha(t) = \omega(\alpha)C(t) + \tilde{m}^\star(t)$, or equivalently, the expression*

$$\frac{d}{dt} \ln |\lambda_\alpha(t) - \tilde{m}^\star(t)|$$

does not depend on α.

This is a rather strong condition. It basically requires that the eigenspaces of D_{Σ_t} coincide for different t up to an overall shift, and that eigenvalues be linearly proportional. Such a property would be very comfortable, but it does not hold for some models of our interest. In particular, it does not hold for the Bianchi I model with distortions. Hence we will derive some of our results under a milder restriction which holds at least for all models where we will assure a time separation exists (see the end of the third chapter).

Definition 1.5 *We will say that the operator D_{Σ_t} has a* **loosely uniform** *spectrum over time if*

$$\left|\frac{d}{dt}\ln|\lambda_\alpha(t) - \tilde{m}^\star(t)|\right| \leq C_{\mathcal{R}},\ \forall t \in \mathcal{R}, \alpha \in \tilde{\Sigma},$$

for any compact interval $\mathcal{R} \subset \mathcal{I}$, and for some $0 < C_{\mathcal{R}} \in \mathbb{R}$ and a smooth function $\tilde{m}^\star(t)$.

If the Fourier transform is conventional, then it will be natural to require that ω be an analytic function on $\tilde{\Sigma}$.

1.4 Some properties of the mode solutions

In this section we investigate the equation (Eq.1.12) and obtain some useful properties of the mode solutions T_α. The mode equation is

$$\ddot{T}_\alpha(t) + F_\alpha(t)\dot{T}_\alpha(t) + G_\alpha(t)T_\alpha(t) = 0,$$

where

$$G_\alpha(t) = H_\alpha(t) + \lambda_\alpha(t),$$

and $\lambda_\alpha(t)$ are the eigenvalues of the operator $D_{\Sigma_t} = -\Delta_t + m^\star(x)$. Note that G_α may become null or negative for some rates of expansion. This corresponds to the so-called positive back-reaction in a linear system and results in exponential solutions. This is an interesting phenomenon appearing in non scalar fields (for scalar fields $H_\alpha = 0$), and its significance is not yet completely clear to us. To understand it one could, for instance, track its influence on the energy-momentum tensor etc. It is not obvious that this is really a physical infrared instability, because it may occur for the co-vector field but not for the vector counterpart, for instance. It is also worth mentioning, that for the co-vector (1-form) field the introduction of a conformal coupling precisely cancels this instability. It seems plausible that for each field there is a choice of the coupling constant which compensates this bad infrared behavior. We say infrared, because $\lambda_\alpha(t)$ attains arbitrarily large positive values at any t, thus on an unbounded subbundle of $R \times \tilde{\Sigma}$, G_α is positive.

We will make this more explicit under the assumption, that D_{Σ_t} has a strictly uniform spectrum. Then the function $C(t)$ is uniformly bounded from below, and the functions H_α and $\tilde{m}^\star$ are uniformly bounded from above on any compact interval $\mathcal{R}$. On the other hand $\omega \to +\infty$, hence is suffices to choose ω large enough to make $G_\alpha = H_\alpha + C\omega + \tilde{m}^\star > 0$.

Fix a component $\tilde{\Sigma}^i$ and write $H = H_\alpha$, $F = F_\alpha$ and $I = I_\alpha$ for all $\alpha \in \tilde{\Sigma}^i$. Define a new variable

$$s(t) = \int_0^t d\tau e^{-\int_0^\tau d\tau' F(\tau')} = \int_0^t d\tau I^{-1}(\tau),$$

which is in a smooth monotone bijective correspondence with t. The inverse function will be denoted by $t(s)$. Regarding all the acting functions of t as functions of s we obtain

$$\ddot{T}_\alpha(s) + \Lambda_\alpha(s) T_\alpha(s) = 0, \tag{1.16}$$

where

$$\Lambda_\alpha(s) = \left[G_\alpha(t) e^{2\int_0^t d\tau F(\tau)} \right]_{t=t(s)} = G_\alpha(s) I^2(s).$$

This is a time dependent harmonic oscillator equation, to which the results in the appendix apply.

Remark 1.5 *Note that the Wronski determinant of two solutions Q, R*

$$\det W[Q,R](s) = Q(s)\dot{R}(s) - \dot{Q}(s)R(s) = const$$

in variable t becomes

$$\det W[Q,R](t) = \frac{dt}{ds}\left(Q(t)\dot{R}(t) - \dot{Q}(t)R(t)\right) = I(t)\left(Q(t)\dot{R}(t) - \dot{Q}(t)R(t)\right) = const.$$

Applying **Corollary 5.1** to (Eq.1.16) for different α we find estimates which in principle depend on α in a complicated way. But under the assumption of loose uniformity on D_{Σ_t} we will be able to invoke more comfortable expressions.

Proposition 1.4 *Suppose D_{Σ_t} has a loosely uniform spectrum over time. Then for a family of arbitrary solutions T_α of (Eq.1.16) the following estimate holds*

$$|T_\alpha(s)| \leq R_{\mathcal{R}} |T_\alpha(0)| + \frac{S_{\mathcal{R}}}{\max\{1, \sqrt{U_{\mathcal{R}} + T_{\mathcal{R}}\lambda_\alpha(0)}\}} |\dot{T}_\alpha(0)|, \; \forall s \in \mathcal{R},$$

with $0 < R_{\mathcal{R}}, S_{\mathcal{R}}, T_{\mathcal{R}} \in \mathbb{R}$ *and* $U_{\mathcal{R}} \in \mathbb{R}$*, for any compact interval* $\mathcal{R}$*.*

Proof: Fix a compact interval $\mathcal{R}$ and for each $\alpha \in \tilde{\Sigma}^i$ apply **Corollary 5.1** with $\Lambda_\alpha(s) = I^2(s)H(s) + I^2(s)\lambda_\alpha(s)$. Because Λ_α is real, we get $A_{\mathcal{R}}(\alpha) = 0$. As $\lambda_\alpha(s)$ is lower semi-bounded we have

$$p_{\mathcal{R}} \doteq \inf_{\tilde{\Sigma}} \inf_{\mathcal{R}} \lambda_\alpha > -\infty.$$

Denote $m_{\mathcal{R}} = \inf_{\mathcal{R}}\{I^2 H\}$ and $n_{\mathcal{R}} = \inf_{\mathcal{R}}\{I^2\} > 0$. Then

$$c_{\mathcal{R}}(\alpha) \geq m_{\mathcal{R}} + n_{\mathcal{R}} \inf_{\mathcal{R}} \lambda_\alpha \geq m_{\mathcal{R}} + n_{\mathcal{R}} p_{\mathcal{R}}.$$

It follows that $\kappa(\alpha) \leq \sqrt{1 + |m_{\mathcal{R}} + n_{\mathcal{R}} p_{\mathcal{R}}|}$ and $e_{\mathcal{R}}(\alpha) \geq 1 + \max\{0, m_{\mathcal{R}} + n_{\mathcal{R}} \inf_{\mathcal{R}} \lambda_\alpha\}$. Denote $M_{\mathcal{R}} = \sup_{\mathcal{R}}\{|I^2 H|\} \geq 0$ and $N_{\mathcal{R}} = \sup_{\mathcal{R}}\{I^2\} > 0$. We find next

$$D_{\mathcal{R}}(\alpha) \leq 1 + |m_{\mathcal{R}} + n_{\mathcal{R}} p_{\mathcal{R}}| + M_{\mathcal{R}} + N_{\mathcal{R}} \left| \sup_{\mathcal{R}} \lambda_\alpha \right|.$$

Now we observe that by loose uniformity

$$\left| \ln \frac{|\lambda_\alpha(s) - \tilde{m}^\star(s)|}{|\lambda_\alpha(s') - \tilde{m}^\star(s')|} \right| = \left| \int_s^{s'} d\sigma \partial_s \ln |\lambda_\alpha(\sigma) - \tilde{m}^\star(\sigma)| \right| \leq |\mathcal{R}| \sqrt{N_{\mathcal{R}}} C_{\mathcal{R}},$$

hence

$$\begin{aligned} \sup_{\mathcal{R}} |\lambda_\alpha - \tilde{m}^\star| &\leq |\lambda_\alpha(0) - \tilde{m}^\star(0)| e^{|\mathcal{R}|\sqrt{N_{\mathcal{R}}} C_{\mathcal{R}}} \\ \inf_{\mathcal{R}} |\lambda_\alpha - \tilde{m}^\star| &\geq |\lambda_\alpha(0) - \tilde{m}^\star(0)| e^{-|\mathcal{R}|\sqrt{N_{\mathcal{R}}} C_{\mathcal{R}}}. \end{aligned} \tag{1.17}$$

Note that whenever $\lambda_\alpha(0) - \tilde{m}^\star(0) > 0$ then it follows by continuity that $\lambda_\alpha(s) - \tilde{m}^\star(s) > 0$ for all $s \in \mathcal{R}$. Denote

$$\lambda_{min} = \tilde{m}^\star(0) - \min\{0, \inf_{\mathcal{R}} \tilde{m}^\star \cdot e^{|\mathcal{R}|\sqrt{N_{\mathcal{R}}} C_{\mathcal{R}}}\} - \min\{0, \frac{m_{\mathcal{R}}}{n_{\mathcal{R}}} e^{|\mathcal{R}|\sqrt{N_{\mathcal{R}}} C_{\mathcal{R}}}\}.$$

Then from $\lambda_\alpha(0) > \lambda_{min}$ it follows $\lambda_\alpha(0) - \tilde{m}^\star(0) > 0$, $m_{\mathcal{R}} + n_{\mathcal{R}} \inf_{\mathcal{R}} \lambda_\alpha > 0$ and

$$\inf_{\mathcal{R}} \lambda_\alpha \geq \inf_{\mathcal{R}} \tilde{m}^\star + (\lambda_\alpha(0) - \tilde{m}^\star(0)) e^{-|\mathcal{R}|\sqrt{N_{\mathcal{R}}} C_{\mathcal{R}}} > 0.$$

Now we have that

$$e_{\mathcal{R}}(\alpha) \geq 1 + \chi[\lambda_\alpha(0) > \lambda_{min}] \left(m_{\mathcal{R}} + n_{\mathcal{R}} (\inf_{\mathcal{R}} \tilde{m}^\star + (\lambda_\alpha(0) - \tilde{m}^\star(0)) e^{-|\mathcal{R}|\sqrt{N_{\mathcal{R}}} C_{\mathcal{R}}}) \right),$$

where the characteristic function χ plays here the role of the condition checking. From (Eq.1.17) we find

$$\sup_{\mathcal{R}} |\lambda_\alpha| \le \sup_{\mathcal{R}} \tilde{m}^\star + |\lambda_\alpha(0) - \tilde{m}^\star(0)| e^{|\mathcal{R}|\sqrt{N_\mathcal{R}} C_\mathcal{R}},$$

whence

$$D_\mathcal{R}(\alpha) \le 1 + |m_\mathcal{R} + n_\mathcal{R} p_\mathcal{R}| + M_\mathcal{R} + N_\mathcal{R}(\sup_{\mathcal{R}} \tilde{m}^\star + |\lambda_\alpha(0) - \tilde{m}^\star(0)| e^{|\mathcal{R}|\sqrt{N_\mathcal{R}} C_\mathcal{R}}).$$

Thus we establish that for $\lambda_\alpha(0) \le \lambda_{min}$

$$\frac{D_\mathcal{R}(\alpha)}{e_\mathcal{R}(\alpha)} \le 1 + |m_\mathcal{R} + n_\mathcal{R} p_\mathcal{R}| + M_\mathcal{R} + N_\mathcal{R}(\sup_{\mathcal{R}} \tilde{m}^\star + (|\lambda_{min}| + |\tilde{m}^\star(0)|) e^{|\mathcal{R}|\sqrt{N_\mathcal{R}} C_\mathcal{R}}),$$

and for $\lambda_\alpha(0) > \lambda_{min}$

$$\frac{D_\mathcal{R}(\alpha)}{e_\mathcal{R}(\alpha)} \le e^{2|\mathcal{R}|\sqrt{N_\mathcal{R}} C_\mathcal{R}} N_\mathcal{R} \left(\frac{1}{n_\mathcal{R}} + \frac{1 + |m_\mathcal{R} + n_\mathcal{R} p_\mathcal{R}| + M_\mathcal{R} + N_\mathcal{R} |\sup_\mathcal{R} \tilde{m}^\star|}{N_\mathcal{R} e^{|\mathcal{R}|\sqrt{N_\mathcal{R}} C_\mathcal{R}}} + \right.$$

$$\left. + \frac{1 + |m_\mathcal{R}| + n_\mathcal{R} |\inf_\mathcal{R} \tilde{m}^\star|}{n_\mathcal{R} e^{-|\mathcal{R}|\sqrt{N_\mathcal{R}} C_\mathcal{R}}} \right).$$

Finally

$$\frac{d}{ds} \ln(\kappa^2 + \Lambda_\alpha) = \frac{\frac{d}{ds}(I^2(H + \tilde{m}^\star)) + \frac{d}{ds}(I^2)(\lambda_\alpha - \tilde{m}^\star) + I^2(\lambda_\alpha - \tilde{m}^\star) \frac{d}{ds} \ln |\lambda_\alpha - \tilde{m}^\star|}{\kappa^2 + I^2 H + I^2 \lambda_\alpha}.$$

Denote $P_\mathcal{R} = \sup_\mathcal{R} |\partial_s(I^2(H - \tilde{m}^\star))| \ge 0$ and $Q_\mathcal{R} = \sup_\mathcal{R} |\partial_s(I^2)| \ge 0$. Again using the loose uniformity, for $\lambda_\alpha - \tilde{m}^\star \le 1$

$$\left| \frac{d}{ds} \ln(\kappa^2 + \Lambda_\alpha) \right| \le P_\mathcal{R} + Q_\mathcal{R} + (N_\mathcal{R})^{\frac{3}{2}} C_\mathcal{R},$$

and else

$$\left| \frac{d}{ds} \ln(\kappa^2 + \Lambda_\alpha) \right| \le \frac{P_\mathcal{R} + Q_\mathcal{R} + (N_\mathcal{R})^{\frac{3}{2}} C_\mathcal{R}}{n_\mathcal{R}}.$$

Summarizing this all we find that by **Corollary 5.1** there exist numbers $0 < R_\mathcal{R}, S_\mathcal{R}, T_\mathcal{R} \in \mathbb{R}$ and $U_\mathcal{R} \in \mathbb{R}$ such that for a family of arbitrary solutions T_α we have

$$|T_\alpha(s)| \le R_\mathcal{R} |T_\alpha(0)| + \frac{S_\mathcal{R}}{\max\{1, \sqrt{U_\mathcal{R} + T_\mathcal{R} \lambda_\alpha(0)}\}} |\dot{T}_\alpha(0)|,$$

what was to be proven. □

The result can be strengthened under additional assumptions. These are perhaps too restrictive, but they appear to be sufficient for some important applications. Let $\mathbb{H}_a = \{z \in \mathbb{C} : |\Im z| < a\}$.

Proposition 1.5 *Suppose the bundle $\mathcal{T}$ is analytic, so that all functions figuring in (Eq.1.16) are real analytic functions of s. Suppose further that D_{Σ_t} has a strictly uniform spectrum. Choose the initial data to be $T_\alpha(0) = p(\omega(\alpha))$ and $\dot{T}_\alpha(0) = q(\omega(\alpha))$, where $p(\omega)$, $q(\omega)$ are holomorphic functions on $\mathbb{H}_a$ for some $a > 0$. Then for each s, $T_\alpha(s) = r_s(\omega(\alpha))$, where $r_s(\omega)$ is holomorphic in ω on $\mathbb{H}_a$ and real analytic in s, and for any compact interval $\mathcal{R}$ it holds*

$$|r_s(\omega)| \leq R_{\mathcal{R}}|p(\omega)| + \frac{S_{\mathcal{R}}}{\max\{1, \sqrt{U_{\mathcal{R}} + T_{\mathcal{R}}\Re\omega}\}}|q(\omega)|,\ \forall s \in \mathcal{R},$$

with $0 < R_{\mathcal{R}}, S_{\mathcal{R}}, T_{\mathcal{R}} \in \mathbb{R}$ and $U_{\mathcal{R}} \in \mathbb{R}$.

Proof: By strict uniformity we have $\lambda_\alpha(t) = \omega(\alpha)C(t) + \tilde{m}^\star(t)$, and if the initial data depend only on ω, then the solutions will also be such. Therefore for convenience we write

$$\ddot{T}_\omega(s) + I^2(s)(H(s) + \omega C(s) + \tilde{m}^\star(s))T_\omega(s) = 0 \tag{1.18}$$

with $T_\omega(0) = p(\omega)$ and $\dot{T}_\omega(0) = q(\omega)$. From the theory of power series it is clear that any real analytic function on $s(\mathcal{I})$ can be extended to a holomorphic function in some open neighborhood $\delta(s(\mathcal{I}))$ of $s(\mathcal{I})$. Consider (Eq.1.18) as a complex differential equation, then for any $\omega \in \mathbb{H}_a$, by Satz 4.1 and Satz 4.2 of [30] there exist neighborhoods $\delta(0)$ of 0 and $\delta(\omega)$ of ω such that $T_\omega(s)$ is holomorphic in $\delta(0) \times \delta(\omega)$. At the same time by Satz 5.3 of [30], for any $\omega \in \mathbb{H}_a$ the solution T_ω can be analytically continued to the whole of $\delta(s(\mathcal{I}))$. Thus $T_\omega(s)$ is holomorphic in $\delta(\mathbb{R}) \times \mathbb{H}_a$. Now restrict back to the real axis and fix the interval $\mathcal{R}$. The reasoning of the previous proposition can be repeated literally except that now $A_{\mathcal{R}}(\alpha)$ is not zero but equals $A_{\mathcal{R}}(\omega) = |\Im\omega| \sup_{\mathcal{R}}\{I^2C\} < a \sup_{\mathcal{R}}\{I^2C\}$. This results in a similar formula as in **Proposition 1.4** with perhaps different coefficients, and that proves our assertion. □

We have an immediate corollary.

Corollary 1.3 *Under the assumptions of* **Proposition 1.5**, *if $p, q \in \mathcal{A}(\mathbb{H}_a)$ then for* *each $s \in \mathcal{R}$, $r_s \in \mathcal{A}(\mathbb{H}_a)$.*

1.5 Mode decomposition of weak solutions

The aim of this section will be to extend the mode decomposition of $Sol_0(\mathcal{T})$ obtained in the previous section to entire $\mathcal{D}(\mathcal{T})'_0$. We will perform it under several natural assumptions, which all will be fulfilled in spatially homogeneous spacetimes discussed later. Here we assume all the conditions of **Proposition 1.3** are satisfied, and we have chosen the system $\{u, v\}_\alpha$ with $u_\alpha = T_\alpha \zeta_\alpha$ and $v_\alpha = \bar{T}_\alpha \zeta_\alpha$, which span $Sol_0(\mathcal{T})$. For convenience we will also assume at least the part (iv) of the definition of the conventional Fourier transform to hold.

Unfortunately we do not have a precise analytical description of the Fourier transformed test function space $\tilde{\mathcal{D}}(\tilde{\Sigma})$ even for the conventional Fourier transform, as it was, for instance, in the Euclidean space by Paley-Wiener theorem. In particular we need to know for which modes T_α it holds

$$T_\alpha(t)\tilde{f}(\alpha) \in \tilde{\mathcal{D}}(\tilde{\Sigma}), \, \forall \tilde{f}(\alpha) \in \tilde{\mathcal{D}}(\tilde{\Sigma}), \, t \in \mathcal{I}. \tag{1.19}$$

At least we are able to find a sufficient condition under additional assumptions.

Proposition 1.6 *Suppose the bundle $\mathcal{T}$ is analytic and D_{Σ_t} has a strictly uniform spectrum. For each $\alpha \in \tilde{\Sigma}^i$ set $T_\alpha(0) = p^i(\omega(\alpha))$ and $\dot{T}_\alpha(0) = q^i(\omega(\alpha))$, where $p^i, q^i \in \mathcal{A}[\mathbb{H}_0]$. Then (Eq.1.19) holds.*

Proof: Choose the interval $\mathcal{R}$ such that it contains both 0 and t. First we note that by **Corollary 1.3** for $\alpha \in \tilde{\Sigma}^i$ we have $T_\alpha(t) = r^i_t(\omega(\alpha))$ with $r^i_t \in \mathcal{A}[\mathbb{H}_0]$. Denote $F^i_t(\lambda) = r^i_t(\frac{\lambda}{C(t)}) \in \mathcal{A}[\mathbb{H}_0]$. Obviously for any $f \in \mathcal{D}(\mathcal{T}_t)$,

$$F^i_t(\lambda_\alpha(t))\tilde{f}(\alpha) = [\widetilde{F^i_t(D_{\Sigma_t})f}](\alpha),$$

where $F^i_t(D_{\Sigma_t})$ is defined by functional calculus. Then by **Proposition 5.4**

$$F^i_t(\lambda_\alpha(t))\tilde{f}(\alpha) \in \tilde{\mathcal{D}}(\tilde{\Sigma}).$$

Let $\{U_n\}$ be a covering by local trivializations of $\mathcal{T}_t$, and let $\{\imath_n\}$ be a subordinate partition of unity. The support of f is covered by N_f (finite) trivializing neighborhoods, and we write $f = \sum_n \imath_n f = \sum_n f_n$. It follows $\tilde{f} = \sum_n \tilde{f}_n$ and $T_\alpha(t)\tilde{f}(\alpha) = \sum_n T_\alpha(t)\tilde{f}_n(\alpha)$. Consider f_n as a section in the trivial bundle $\pi^{-1}(U_n)$. As we have seen already (and as we will see even more evidently for homogeneous spacetimes in the next chapter) each component $\tilde{\Sigma}^i$ supports the Fourier transform of one fiber component in some local frame. Thus we can write $f_n = \sum_i f_n^i$, where $f_n^i \in \mathcal{D}(U_n)$ and $\tilde{f}_n^i$ is supported in $\tilde{\Sigma}^i$. We get

$$T_\alpha(t)\tilde{f}(\alpha) = \sum_n \sum_i F_t^i(\lambda_\alpha(t))\tilde{f}_n^i(\alpha) = \sum_n \sum_i [\widetilde{F_t^i(D_{\Sigma_t})f_n^i}](\alpha) \in \tilde{\mathcal{D}}(\tilde{\Sigma}),$$

which completes the proof. □

Remark 1.6 *An argument involving local trivializations as in the proof of* **Proposition 1.6** *will show that the multiplication of $\mathcal{F}[f(t,.)]$ by $I_\alpha(t)$ amounts to multiplication of each fiber component by a number, hence $I_\alpha(t)\mathcal{F}[f(t,.)] \in \tilde{\mathcal{D}}(\tilde{\Sigma})$ for all $t \in \mathcal{I}$.*

Two useful facts about the time dependent Fourier transform can be given by the following

Proposition 1.7 *Let $\tilde{f}(t,\alpha) \in C^0_\infty\left(\mathcal{I}, \tilde{\mathcal{D}}(\tilde{\Sigma})\right)$. then*

(i) $f(t,\vec{x}) = \mathcal{F}^{-1}[\tilde{f}(t,\alpha)] \in \mathcal{D}(\mathcal{T})$

(ii) $\int_{\mathcal{I}} dt\tilde{f}(t,\alpha) \in \tilde{\mathcal{D}}(\tilde{\Sigma})$.

Proof: Let

$$f(t,\vec{x}) = \mathcal{F}^{-1}[\tilde{f}(t,\alpha)] = \int_{\tilde{\Sigma}} d\mu(\alpha)\tilde{f}(t,\alpha)\zeta_\alpha(\vec{x}).$$

For each $t \in \mathcal{I}$ we have $\tilde{f}(t,\alpha) \in \tilde{\mathcal{D}}(\tilde{\Sigma})$ and therefore $f(t,\vec{x}) \in \mathcal{D}(\mathcal{T}_t)$. If the compact interval $A \subset \mathcal{I}$ is such that $\forall t \notin A$, $\tilde{f}(t,\alpha) = 0$, then obviously $\forall t \notin A$, $f(t,\vec{x}) = 0$. Because the integration converges in $L^2(\tilde{\Sigma},\mu)$, differentiation can be interchanged with the integral, thus $f(t,\vec{x})$ is smooth in t. The part (i) is proven.

Now write

$$\tilde{f}(t,\alpha) = \int_{\Sigma_t} d\mu_{h_0}(\vec{x})\langle\bar{\zeta}_\alpha(\vec{x}), f(t,\vec{x})\rangle_{\mathfrak{g}^0},$$

and

$$\int_{\mathcal{I}} dt\tilde{f}(t,\alpha) = \int_{\mathcal{I}} dt \int_{\Sigma_t} d\mu_{h_0}(\vec{x})\langle\bar{\zeta}_\alpha(\vec{x}), f(t,\vec{x})\rangle_{\mathfrak{g}^0} =$$

$$= \int_{\Sigma_t} d\mu_{h_0}(\vec{x})\langle\bar{\zeta}_\alpha(\vec{x}), \int_{\mathcal{I}} dt f(t,\vec{x})\rangle_{\mathfrak{g}^0} = \mathcal{F}[\int_{\mathcal{I}} dt f(t,\vec{x})],$$

where Fubini's theorem was used with the justification that both integrals run over compact supports [16]. For the part (ii) it remains to show that $\int_{\mathcal{I}} dt f(t,\vec{x}) \in \mathcal{D}(\mathcal{T}_t)$. But this is again clear because the integral runs over a compact support. □

Next we want to show that the Cauchy problem can be well-posed in the distributional sense. We will do it by generalizing (Eq.1.1) to distributional solutions.

Proposition 1.8 *For any $u_0, u_1 \in \mathcal{D}(\mathcal{T}_t)'$ there exists a unique $\jmath(u_0,u_1) = u \in \mathcal{D}(\mathcal{T})'_0$ such that*

$$u(f) = u_0(i_t^*(\nabla_t E[f])) - u_1(i_t^*(E[f])),\ \forall f \in \mathcal{D}(\mathcal{T}).$$

Proof: By **Proposition 1.1** we know that E is surjective, so we denote the bijective part of E to be $E_\updownarrow : \mathcal{D}(\mathcal{T})/\ker E \to Sol_0(\mathcal{T})$. For surjectivity of $\jmath$ it suffices to set

$$u_0(v_1) = u(E_\updownarrow^{-1}[\jmath(0,v_1)]),\ u_1(v_0) = -u(E_\updownarrow^{-1}[\jmath(v_0,0)]),\ \forall v_0, v_1 \in \mathcal{D}(\mathcal{T}_t).$$

Indeed,

$$u(f) = u(E_\updownarrow^{-1}[E_\updownarrow[f]]) = u(E_\updownarrow^{-1}[\jmath(i_t^*(E_\updownarrow[f]), i_t^*(\nabla_t E_\updownarrow[f]))]) =$$

$$= u(E_\updownarrow^{-1}[\jmath(i_t * (E_\updownarrow[f]),0)]) + u(E_\updownarrow^{-1}[\jmath(0, i_t^*(\nabla_t E_\updownarrow[f]))]) = u_0(i_t^*(\nabla_t E[f])) - u_1(i_t^*(E[f])).$$

For injectivity of $\jmath$ let $u_0, u_1 \in \mathcal{D}(\mathcal{T}_t)'$ be given. Define u as in the statement. Then obviously $u(Df) = 0$ because $EDf = 0$ for any $f \in \mathcal{D}(\mathcal{T})$, hence $u \in \mathcal{D}(\mathcal{T}_t)'_0$. Now suppose the same formula holds also for different $u_0', u_1' \in \mathcal{D}(\mathcal{T}_t)'$ with the same u. Then we have

$$0 = (u_0 - u_0')(i_t^*(\nabla_t E[f])) - (u_1 - u_1')(i_t^*(E[f])),\ \forall f \in \mathcal{D}(\mathcal{T}).$$

Evaluating on $f = E_\updownarrow^{-1}(\jmath(v_0,0))$ and $g = E_\updownarrow^{-1}(\jmath(0,v_1))$ for arbitrary $v_0, v_1 \in \mathcal{D}(\mathcal{T}_t)$ we find $u_0 = u_0'$ and $u_1 = u_1'$. □

Now we come to the main assertion. Let the modes T_α be chosen such that (Eq.1.19) holds.

Proposition 1.9 *Under the assumptions made, there exist closed topological subspaces* $\tilde{\mathcal{D}}^u(\tilde{\Sigma}), \tilde{\mathcal{D}}^v(\tilde{\Sigma}) \subset \tilde{\mathcal{D}}(\tilde{\Sigma})$*, such that for any* $\psi \in \mathcal{D}(\mathcal{T})'_0$ *there are unique distributions* $a^\psi \in \tilde{\mathcal{D}}^u(\tilde{\Sigma})'$*,* $b^\psi \in \tilde{\mathcal{D}}^v(\tilde{\Sigma})'$ *with*

$$\psi(f) = a^\psi(u_\alpha(f)) + b^\psi(v_\alpha(f)),\ \forall f \in \mathcal{D}(\mathcal{T}).$$

Proof: Considered as distributions, the functions u_α act as

$$u_\alpha(f) = \langle u_\alpha, f\rangle_M = \int_{\mathcal{I}} dt T_\alpha(t)\langle\zeta_\alpha, f(t,.)\rangle_{\Sigma_t} = \int_{\mathcal{I}} dt T_\alpha(t)(\zeta_{-\alpha}, \check{\Gamma} f(t,.))_{\Sigma_t} =$$

$$= \int_{\mathcal{I}} dt T_\alpha(t) I_\alpha(t) \mathcal{F}[\check{\Gamma} f(t,.)](-\alpha),\ \forall f \in \mathcal{D}(\mathcal{T}).$$

(Remember that $g_{00} = 1$.) The action of v_α is similar. By assumption (Eq.1.19) and **Remark 1.6** we find

$$T_\alpha(t) I_\alpha(t) \mathcal{F}[\check{\Gamma} f(t,.)](-\alpha) \in \tilde{\mathcal{D}}(\tilde{\Sigma}).$$

Then by **Proposition 1.7** we get $u_\alpha(f) \in \tilde{\mathcal{D}}(\tilde{\Sigma})$ (similarly for v_α).

In general, the maps $f \to u_\alpha(f)$ and $f \to v_\alpha(f)$ need not be surjective. Therefore we define

$$\tilde{\mathcal{D}}^u(\tilde{\Sigma}) = u_\alpha(\mathcal{D}(\mathcal{T})).$$

By continuity of the map $f \to u_\alpha(f)$ (which is easy to establish), $\tilde{\mathcal{D}}^u(\tilde{\Sigma})$ is a closed subspace of $\tilde{\mathcal{D}}(\tilde{\Sigma})$. Similarly we define $\tilde{\mathcal{D}}^v(\tilde{\Sigma})$.

Recall the mode expansion for arbitrary $\phi \in Sol_0(\mathcal{T})$,

$$\phi(t, \vec{x}) = \int_{\tilde{\Sigma}} d\mu(\alpha) \left[a^\phi_\alpha u_\alpha(x) + b^\phi_\alpha v_\alpha(x)\right].$$

Thus $Sol_0(\mathcal{T})$ can be written as a direct sum of linear subspaces, $Sol_0(\mathcal{T}) = Sol^u_0(\mathcal{T}) \oplus Sol^v_0(\mathcal{T})$, with

$$Sol^u_0(\mathcal{T}) = \{\phi \in Sol_0(\mathcal{T})\colon\ b^\phi = 0\},\ Sol^v_0(\mathcal{T}) = \{\phi \in Sol_0(\mathcal{T})\colon\ a^\phi = 0\},$$

and we will write $\phi = \phi^u + \phi^v$. Regarding as a distribution in $\mathcal{D}(\mathcal{T})'$, ϕ^u and ϕ^v act as

$$\phi^u(f) = \int_{\tilde{\Sigma}} d\nu(\alpha) a^\phi_\alpha u_\alpha(f),\ \phi^v(f) = \int_{\tilde{\Sigma}} d\nu(\alpha) b^\phi_\alpha v_\alpha(f).$$

The functions $a_\alpha^\phi, b_\alpha^\phi$ can be regarded as distributions $a^\phi \in \tilde{\mathcal{D}}^u(\tilde{\Sigma})'$, $b^\phi \in \tilde{\mathcal{D}}^v(\tilde{\Sigma})'$, and we can write

$$\phi(f) = \phi^u(f) + \phi^v(f) = a^\phi(u_\alpha(f)) + b^\phi(v_\alpha(f)). \tag{1.20}$$

Now let $\varphi \in Sol(\mathcal{T})$ be a solution, which does not necessarily have $supp\{\varphi\} \cap \Sigma_t$ compact. Its Cauchy data are

$$(i_t^*(\varphi), i_t^*(\nabla_t \varphi)) = (\varphi_0, \varphi_1) \in \mathcal{E}(\mathcal{T}_t) \oplus \mathcal{E}(\mathcal{T}_t).$$

Choosing a countable (compactly finite) partition of unity on Σ we can write

$$(\varphi_0, \varphi_1) = \sum_{i=1}^{\infty} (\phi_0^i, \phi_1^i),\ (\phi_0^i, \phi_1^i) \in \mathcal{D}(\mathcal{T}_t) \oplus \mathcal{D}(\mathcal{T}_t),$$

where the sum involves finite items on any compact region $U \in \Sigma$. Now for each i we have

$$\phi^i = i_t^{-1}(\phi_0^i, \phi_1^i) \in Sol_0(\mathcal{T}),$$

thus

$$\phi^i = \phi^{i,u} + \phi^{i,v} = i_t^{-1}(\phi_0^{i,u}, \phi_1^{i,u}) + i_t^{-1}(\phi_0^{i,v}, \phi_1^{i,v}),\ \phi^{i,u} \in Sol_0^u(\mathcal{T}),\ \phi^{i,v} \in Sol_0^v(\mathcal{T}).$$

Set

$$\varphi^u = \sum_{i=1}^{\infty} i_t^{-1}(\phi_0^{i,u}, \phi_1^{i,u}) = \sum_{i=1}^{\infty} \phi^{i,u},$$

and

$$\varphi^v = \sum_{i=1}^{\infty} i_t^{-1}(\phi_0^{i,v}, \phi_1^{i,v}) = \sum_{i=1}^{\infty} \phi^{i,v},$$

where the sums converge in compact topology. (This can be seen as follows. The intersection of the causal cone of any compact region with a Cauchy surface is a compact surface, and therefore only finite summands survive.) But we have

$$\phi^{i,u}(f) = a^{\phi^i}(u_\alpha(f)),\ \phi^{i,v}(f) = b^{\phi^i}(v_\alpha(f))$$

for some distributions a^{ϕ^i} and b^{ϕ^i}. Thus we obtain

$$\varphi^u(f) = \sum_{i=1}^{\infty} \phi^{i,u}(f) = \sum_{i=1}^{\infty} a^{\phi^i}(u_\alpha(f))$$

and

$$\varphi^v(f) = \sum_{i=1}^{\infty} \phi^{i,v}(f) = \sum_{i=1}^{\infty} b^{\phi^i}(v_\alpha(f)).$$

This convergence defines distributions

$$a^\varphi = \sum_{i=1}^{\infty} a^{\phi^i} \in \tilde{\mathcal{D}}^u(\tilde{\Sigma})', \, b^\varphi = \sum_{i=1}^{\infty} b^{\phi^i} \in \tilde{\mathcal{D}}^v(\tilde{\Sigma})',$$

such that

$$\varphi^u(f) = a^\varphi(u_\alpha(f)), \, \varphi^v(f) = b^\varphi(v_\alpha(f)), \, \varphi = \varphi^u + \varphi^v, \, \forall f \in \mathcal{D}(\mathcal{T}),$$

and thus $Sol(\mathcal{T}) = Sol^u(\mathcal{T}) \oplus Sol^v(\mathcal{T})$, where

$$Sol^u(\mathcal{T}) = \{\varphi \in Sol(\mathcal{T}) \colon \, b^\varphi = 0\}, \, Sol^v(\mathcal{T}) = \{\varphi \in Sol(\mathcal{T}) \colon \, a^\varphi = 0\}.$$

Now let $\psi \in \mathcal{D}(\mathcal{T})'_0$ be a weak solution, and $\{\chi_m\}$ a usual mollifier on Σ_t. Define the mollifications $\heartsuit_m \psi \in Sol(\mathcal{T})$ by

$$\heartsuit_m \psi = \jmath(\chi_m \psi_0, \chi_m \psi_1),$$

where $\psi = \jmath(\psi_0, \psi_1)$ by **Proposition 1.8**. Then it is easy to see that $\heartsuit_m \psi \to \psi$ in $\mathcal{D}(\mathcal{T})'$. That $\heartsuit_m \psi \in Sol(\mathcal{T})$ it follows

$$\heartsuit_m \psi = (\heartsuit_m \psi)^u + (\heartsuit_m \psi)^v, \, (\heartsuit_m \psi)^\bullet \in Sol^\bullet(\mathcal{T}).$$

The disjointness $Sol^u(\mathcal{T}) \cap Sol^v(\mathcal{T}) = 0$ implies that $(\heartsuit_m \psi)^u \to \psi^u$ and $(\heartsuit_m \psi)^v \to \psi^v$ with some distributions $\psi^u \in \overline{Sol^u(\mathcal{T})}$, $\psi^v \in \overline{Sol^v(\mathcal{T})}$, such that $\psi = \psi^u + \psi^v$. We denote

$$\psi^u(f) = \lim_{m\to\infty} (\heartsuit_m \psi)^u(f) = \lim_{m\to\infty} a^{\psi_m}(u_\alpha(f)) \doteq a^\psi(u_\alpha(f)),$$

$$\psi^v(f) = \lim_{m\to\infty} (\heartsuit_m \psi)^v(f) = \lim_{m\to\infty} b^{\psi_m}(v_\alpha(f)) \doteq b^\psi(v_\alpha(f)),$$

for some distributions a^ψ and b^ψ. Finally we arrive at

$$\psi(f) = a^\psi(u_\alpha(f)) + b^\psi(v_\alpha(f)), \, \forall f \in \mathcal{D}(\mathcal{T}).$$

The map

$$\mathcal{D}(\mathcal{T})'_0 \ni \psi \to (a^\psi, b^\psi) \in \tilde{\mathcal{D}}^u(\tilde{\Sigma})' \oplus \tilde{\mathcal{D}}^v(\tilde{\Sigma})'$$

is a bijection by construction. □

1.6 The propagator

In this section we will find the explicit form of the propagator E in terms of the mode decomposition. Of course, Green's functions can be calculated using the techniques of inverse operators. But our approach will be more concordant to the spirit of this work and will at the same time demonstrate the usefulness of the mode decomposition in general.

To use the mode decomposition for weak solutions we assume that at least the condition (iv) of the conventional Fourier transform holds, and that the assumptions of **Proposition 1.9** are satisfied. Choose mode solutions to be such that $T_\alpha(0) = T_{-\alpha}(0)$ and $\dot{T}_\alpha(0) = \dot{T}_{-\alpha}(0)$. Then because $\alpha \to -\alpha$ preserves both $\lambda_\alpha(t)$ and the component $\tilde{\Sigma}^i$, we have the same mode equations for T_α and $T_{-\alpha}$, hence everywhere $T_\alpha(t) = T_{-\alpha}(t)$.

The function

$$\det W[T_\alpha, \bar{T}_\alpha](t) = I_\alpha(t) \left[\dot{T}_\alpha(t)\bar{T}_\alpha(t) - T_\alpha(t)\dot{\bar{T}}_\alpha(t)\right] \in C^\infty(\mathcal{I}, i \cdot \mathbb{R})$$

is the Wronskian of two independent solutions T_α and $\bar{T}_\alpha$ and is therefore an imaginary constant. For convenience we appoint once and forever to consider only the modes normalized by

$$\dot{T}_\alpha(t)\bar{T}_\alpha(t) - T_\alpha(t)\dot{\bar{T}}_\alpha(t) = i \cdot I_\alpha^{-1}(t). \tag{1.21}$$

It can be seen that this condition is consistent with our previous assumptions for the modes T_α.

We remark that the Krein space involution $\check{\Gamma}$ commutes with the connection components Γ_i. Indeed, by definition $\check{\Gamma} = P^+ - P^-$, where $P^\pm$ are the projections onto the subspaces of positive/negative definiteness of the metric $\langle , \rangle_g$. Let $\{e_i\}$ be a pseudo-orthonormal moving frame, i.e., $\langle e_i, e_i \rangle_g = \pm 1$. The value of each $\langle e_i, e_i \rangle_g$ is preserved under ∇, and therefore e_i remains in the same eigenspace of $\check{\Gamma}$, although in our main frame e_i

experiences gradient,

$$\nabla e_i = \sum_{j=1}^{4}\sum_{k=1}^{n}\Gamma^k_{ji}dx^j \otimes e_k.$$

Hence $\check{\Gamma}$ commutes with all Γ_i. We have that

$$\langle u, v\rangle_{\Sigma_t} = (\check{\Gamma}\bar{u}, v)_{\Sigma_t} = \int_{\tilde{\Sigma}} d\mu(\alpha)s(\alpha)I_\alpha(t)\tilde{u}(-\alpha)\tilde{v}(\alpha),\ \forall u \in \mathcal{E}(\mathcal{T}_t),\ v \in \mathcal{D}(\mathcal{T}_t),$$

where $s(\alpha)$ is the Fourier image of the Krein involution $\check{\Gamma}$, which due to the remark above satisfies $s(\tilde{\Sigma}^i) = \{+1, -1\}$, i.e., is constant on each component $\tilde{\Sigma}^i$. We have used the fact that $\tilde{\bar{\bar{u}}}(\alpha) = \tilde{u}(-\alpha)$ which follows from the condition (iv) of the conventional Fourier transform.

Now the propagator is the unique operator $E : \mathcal{D}(\mathcal{T}) \to Sol_0(\mathcal{T})$ which satisfies

$$v(f) = \langle v(t;.), \dot{E}[f](t;.)\rangle_{\Sigma_t} - \langle \dot{v}(t;.), E[f](t;.)\rangle_{\Sigma_t},\ \forall v \in Sol_0(\mathcal{T}),\ f \in \mathcal{D}(\mathcal{T}),\ t \in \mathcal{I}. \tag{1.22}$$

As $v \in Sol_0(\mathcal{T})$ we can write

$$v(x) = \int_{\tilde{\Sigma}} d\mu(\alpha)a^v(\alpha)u_\alpha(x) + \int_{\tilde{\Sigma}} d\mu(\alpha)b^v(\alpha)v_\alpha(x), \tag{1.23}$$

and

$$\widetilde{v(t;.)}(\alpha) = a^v(\alpha)T_\alpha(t) + b^v(\alpha)\bar{T}_\alpha(t),$$
$$\widetilde{\dot{v}(t;.)}(\alpha) = a^v(\alpha)\dot{T}_\alpha(t) + b^v(\alpha)\dot{\bar{T}}_\alpha(t).$$

Similarly for $E[f] \in Sol_0(\mathcal{T})$,

$$E[f](x) = \int_{\tilde{\Sigma}} d\mu(\alpha)a^E[f](\alpha)u_\alpha(x) + \int_{\tilde{\Sigma}} d\mu(\alpha)b^E[f](\alpha)v_\alpha(x),$$

$$\widetilde{E[f](t;.)}(\alpha) = a^E[f](\alpha)T_\alpha(t) + b^E[f](\alpha)\bar{T}_\alpha(t),$$

and

$$\widetilde{\dot{E}[f](t;.)}(\alpha) = a^E[f](\alpha)\dot{T}_\alpha(t) + b^E[f](\alpha)\dot{\bar{T}}_\alpha(t),$$

with some distribution fields $a^E[f](\alpha)$ and $b^E[f](\alpha)$. Using all this we compute

$$\langle v(t;.), \dot{E}[f](t;.)\rangle_{\Sigma_t} - \langle \dot{v}(t;.), E[f](t;.)\rangle_{\Sigma_t} =$$

$$= -\int_{\tilde{\Sigma}} d\mu(\alpha)s(\alpha)I_\alpha(t)\left[\dot{T}_\alpha(t)\bar{T}_\alpha(t) - T_\alpha(t)\dot{\bar{T}}_\alpha(t)\right]\left[a^v(-\alpha)b^E[f](\alpha) - b^v(-\alpha)a^E[f](\alpha)\right] =$$

by normalization (Eq.1.21)

$$= -i \int_{\tilde{\Sigma}} d\mu(\alpha) s(\alpha) I_\alpha(t) \left[a^v(-\alpha) b^E[f](\alpha) - b^v(-\alpha) a^E[f](\alpha) \right]. \tag{1.24}$$

On the other hand, we know that $ED = 0$, thus $a^E[f](\alpha)$ and $b^E[f](\alpha)$ are weak solutions of the field equation and can be mode decomposed as

$$\begin{aligned} a^E[f](\alpha) &= a^1_\alpha(u_\beta(f)) + a^2_\alpha(v_\beta(f)), \\ b^E[f](\alpha) &= b^1_\alpha(u_\beta(f)) + b^2_\alpha(v_\beta(f)). \end{aligned} \tag{1.25}$$

By (Eq.1.23) we have

$$v(f) = \int_{\tilde{\Sigma}} d\mu(\alpha) a^v(\alpha) u_\alpha(f) + \int_{\tilde{\Sigma}} d\mu(\alpha) b^v(\alpha) v_\alpha(f). \tag{1.26}$$

Inserting (Eq.1.24), (Eq.1.25) and (Eq.1.26) into (Eq.1.22) we obtain

$$a^1_\alpha = b^2_\alpha = 0,\ a^2_\alpha = -b^1_\alpha = i \cdot s(\alpha) \delta(\beta - \alpha).$$

And our final formula is

$$E[f](x) = i \int_{\tilde{\Sigma}} d\mu(\alpha) s(\alpha) \left[v_{-\alpha}(f) u_\alpha(x) - u_{-\alpha}(f) v_\alpha(x) \right],$$

which is in full accord with the result obtained by [32] for scalar fields on FRW spacetimes.

Chapter 2

Aspects of harmonic analysis in homogeneous spacetimes

2.1 Spatially Homogeneous Cosmological Models

The main goal of the current work is to refurbish the mathematical framework of quantum field theory on classical cosmological spacetimes, in general, and to advance towards a satisfactory rigorous description of cosmological particle creation in states of low energy for hyperbolic fields, in particular. The latter would be an extension of results obtained in [14] for the Klein-Gordon field on specific FRW models to more general situations. Thus although some results were and will be obtained under abstract general assumptions, our attention is concentrated at the geometrical setup of most common cosmological models. Supported by observations of the universe at large scale, cosmology considers mainly spatially homogeneous, or in addition also isotropic, spacetimes. A condensed account of cosmological arguments and their geometrical implications can be found, for instance, in [39]. The essence of these geometrical restrictions is mathematically expressed by imposing the existence of a sufficiently rich system of symmetries (more precisely, a group of spatial isometries) on the spacetime. Extensive treatments of all possible isometry groups and related questions can be found in [39], [45], [53]. An introduction to the generalities of harmonic analysis on vector bundles is given in [7]. In this section we will try to deductively introduce our geometrical setup with the help of the information in the above mentioned references.

Foliation by equal time Cauchy hypersurfaces. Recall that we are working with a 4-dimensional globally hyperbolic Lorentzian manifold (M, g) on which a global smooth time function and an atlas can be chosen following [3] such that M is foliated by 3-dimensional spacelike equal-time smooth Cauchy hypersurfaces and

$$ds^2 = g_{00}dt^2 - d\sigma^2,$$

where $d\sigma^2$ is the line element on any of those Cauchy surfaces being Riemannian submanifolds.

The structure group. Any vector bundle $\mathcal{T}$ can be considered as associated to its frame bundle $\mathcal{P}_\mathcal{T}$ with structure group $GL(n)$. If we want the fiberwise transformations to respect the fiber metric, then we have to restrict the principal bundle to the orthogonal frame bundle. All fibers V_p with their respective non-degenerate pseudo-Riemannian structures $\mathfrak{g}_p$ are isomorphic, and their generalized orthogonal groups $O(\mathfrak{g}_p)$ (i.e., groups of invertible endomorphisms of V_p preserving $\mathfrak{g}_p$) are isomorphic to the generalized Lorentz group $O(\pm_\mathfrak{g})$, where $\pm_\mathfrak{g}$ in this context will be understood as the signature of $\mathfrak{g}$. But the same vector bundle $\mathcal{T}$ can be associated also to another principal bundle (which we again denote by $\mathcal{P}_\mathcal{T}$) with structure group H (say, for field theoretical reasons). Then we have a representation r of H on V. If r also respects the metric, then $r(H) \in O(\pm_\mathfrak{g})$, so H is homomorphic to $O(\pm_\mathfrak{g})$. For instance, $H = SO^+(\pm_\mathfrak{g})$ (tensor bundle) or $H = Spin^+(\pm_\mathfrak{g})$ (spinor bundle).

Isometries. Let us start with reminding some definitions. An *isometry* of the spacetime (M, g) is a diffeomorphism $\psi : M \to M$ such that $\psi^* g = g$ holds on M, where ψ^* is the pullback of ψ. If $\psi' : M \to M$ is another isometry, then obviously such is also their superposition $\psi \circ \psi'$. With the superposition as product, isometries thus constitute an abstract group, which we will denote $\mathbf{Iso}(M)$. If $\mathcal{T} \to M$ is a the vector bundle over M as defined previously, then an *isometry* of the vector bundle $\mathcal{T}$ is a morphism $\Psi : \mathcal{T} \to \mathcal{T}$ covering an isometry of the base, $\pi \circ \Psi \circ \pi^{-1} \in \mathbf{Iso}(M)$, such that $\Psi^* \mathfrak{g} = \mathfrak{g}$ and $\Psi^* \nabla = \nabla$ (or more precisely $\Psi^* D = D$ when a normal hyperbolic field operator D is specified), where Ψ^* denotes pullback maps, $\mathfrak{g}$ is the pseudo-Riemannian fiber metric, and ∇ is the metric connection. Again via superposition, the isometries of the bundle $\mathcal{T}$ comprise an abstract group $\mathbf{Iso}(\mathcal{T})$.

The map $\mathbf{Iso}(\mathcal{T}) \ni \Psi \to \pi \circ \Psi \circ \pi^{-1} \in \mathbf{Iso}(M)$ gives a homomorphism of $\mathbf{Iso}(\mathcal{T})$ into $\mathbf{Iso}(M)$. The image of this homomorphosm is a subgroup of $\mathbf{Iso}(M)$ and will be denoted by $\mathbf{Iso}^{\mathcal{T}}(M) \subset \mathbf{Iso}(M)$, and its kernel is a normal subgroup of $\mathbf{Iso}(\mathcal{T})$. This kernel $\mathbf{Iso}(\mathcal{T})/\mathbf{Iso}^{\mathcal{T}}(M)$ consists of isometries of the bundle $\mathcal{T}$ covering the identity map of M. These are precisely the smooth sections in the principle bundle $\mathcal{P}_{\mathcal{T}} \stackrel{loc}{\longrightarrow} M \times H$ of $\mathcal{T}$, i.e., $\mathbf{Iso}(\mathcal{T})/\mathbf{Iso}^{\mathcal{T}}(M) = C^{\infty}(\mathcal{P}_{\mathcal{T}})$. The group multiplication is given by the pointwise multiplication of sections.

Homogeneous bundle structure. If the sections in the bundle $\mathcal{T}$ are going to represent physical fields, than one should have a concrete picture of how they transform under the diffeomorphisms of the spacetime M. In case of the tensor bundle this picture is automatically encoded in the pullback map. An abstract vector bundle does not have such a structure by itself. Thus a physical field theory has to specify a homomorphism $\rho : \mathrm{Diff}(M) \to C^{\infty}(\mathcal{P}_{\mathcal{T}})$. For the tangent bundle $\rho(\psi) = d\psi$, $\psi \in \mathrm{Diff}(M)$. When considering arbitrary diffeomorphism, then the structure group should be $GL(n)$ rather than a smaller H. But if we restrict ρ to $\rho : \mathbf{Iso}^{\mathcal{T}}(M) \to C^{\infty}(\mathcal{P}_{\mathcal{T}})$, then H can be chosen. For brevity denote $G = \mathbf{Iso}^{\mathcal{T}}(M)$. We have the injection

$$G \ni g \to g \times \rho(g) \in \mathbf{Iso}(\mathcal{T}),$$

which gives sense to the left action of G on $\mathcal{T}$ by isometries.

The abstract group of isometries of a pseudo-Riemannian manifold of dimension m is given the compact open topology, in which it becomes a Lie group of dimension at most $n(n+1)/2$ [29]. It can be further shown, that the compact open topology in this case is equivalent to the pointwise convergence topology of isometries. Thus we automatically obtain a Lie group structure on $\mathbf{Iso}(M)$. Then $G \subset \mathbf{Iso}(M)$ is a topological subgroup defined by

$$G = \{\psi \in \mathbf{Iso}(M)\colon\ (\psi \times \rho(\psi))^{*}\, \mathfrak{g} = \mathfrak{g},\ (\psi \times \rho(\psi))^{*}\, D = D\}.$$

If ρ is a continuous homomorphism, then all the operations in the equations

$$(\psi \times \rho(\psi))^{*}\, \mathfrak{g} = \mathfrak{g},\ (\psi \times \rho(\psi))^{*}\, D = D$$

are continuous, and therefore the subspace G of $\mathbf{Iso}(M)$ defined by this equation is a closed topological subspace. But then by Cartan's theorem G is actually a Lie subgroup, as it is a closed topological subgroup of the Lie group $\mathbf{Iso}(M)$. Thus we have the structure of a G-homogeneous vector bundle $\mathcal{T}$.

Spatially homogeneous bundle. The bundle $\mathcal{T}$ will be called *spatially homogeneous* if the orbits of $\mathbf{Iso}^{\mathcal{T}}(M)$ are 3-dimensional smooth spacelike hypersurfaces which foliate M. (Maybe it is worth mentioning here that everywhere in this work we consider only connected spacetimes M.) By **Theorem 8.16** in [53] there exists a parametrization of these orbits by the affine parameter of the family of normal geodesics, such that the metric takes the form

$$ds^2 = dt^2 - d\sigma^2.$$

On the other hand, our original foliation by equal time Cauchy surfaces due to **Theorem 1.1** in [3] also yielded such a metric form. We assume that the time function can be chosen such that equal time Cauchy surfaces are the orbits of $\mathbf{Iso}^{\mathcal{T}}(M)$ (probably this can be shown to be true in general). We note that due to the transitive action of G on Σ_t for every t, it holds $G \subset \mathbf{Iso}^{\mathcal{T}_t}(\Sigma_t)$. We did not write $G = \mathbf{Iso}^{\mathcal{T}_t}(\Sigma_t)$ because it is possible that for some $t \neq t' \in \mathcal{I}$, $\mathbf{Iso}^{\mathcal{T}_t}(\Sigma_t) \neq \mathbf{Iso}^{\mathcal{T}_{t'}}(\Sigma_{t'})$, i.e., fore some time the time slice may be more symmetric than usual. We will concentrate on G, which is the maximal guaranteed amount of symmetry which is present at any time. Thus we see that $\mathcal{T}_t$ also has the structure of a G-homogeneous vector bundle.

Consider the principle bundle $\mathcal{P}_{\mathcal{T}_t}$ of $\mathcal{T}_t$, which is a subbundle of $\mathcal{P}_{\mathcal{T}}$. The smooth left action of G on $\mathcal{T}_t$ gives a smooth left action of G on $\mathcal{P}_{\mathcal{T}_t}$ as well. This action allows one to construct a global smooth section in $\mathcal{P}_{\mathcal{T}_t}$, whence it follows that the bundle $\mathcal{T}_t$ is trivial. Because $M \sim \Sigma_t \times \mathcal{I}$, the whole bundle $\mathcal{T}$ is also trivial. Thus spatially homogeneous vector bundles over M are necessarily trivial.

The requirement that the field operator D is G-invariant implies that the function $m^\star(x)$ is in fact a function of time only.

Homogeneous space structure. Now let $\mathbf{StabIso}^{\mathcal{T}}(p) \subset G$ be the stabilizer of G at some fixed point $p \in M$. Then $\mathbf{StabIso}^{\mathcal{T}}(p)$ is a closed Lie subgroup by Cartan's theorem. If for all $p \in M$, the groups $\mathbf{StabIso}^{\mathcal{T}}(p)$ are isomorphic, then we denote them all by $\mathbf{StabIso}^{\mathcal{T}}(M)$. In this case the orbits Σ_t of G are diffeomorphic to the homogeneous space $G/\mathbf{StabIso}^{\mathcal{T}}(M) \doteq \Sigma$. Denote $O = \mathbf{StabIso}^{\mathcal{T}}(M)^+$, the identity component. Then $\Gamma = \mathbf{StabIso}^{\mathcal{T}}(M)/O$ is a discrete normal subgroup of G. If the homogeneous space Σ is itself a Lie subgroup of G, then it acts on each Σ_t simply transitively.

The 4-dimensional reality. As already mentioned, the isometry group $\mathbf{Iso}(M)$ of the $n = 4$ dimensional spacetime M is a Lie group of dimension at most $n(n+1)/2 =$

10. Thus in principle one can construct all real Lie algebras $\mathcal{G}$ of dimension up to 10, their corresponding connected simply connected Lie groups G, then all discrete normal subgroups Γ of such G etc., thereby exhausting all possible isometry groups of M. This heavy task have been done by Petrov et al [45] and others [53], and all the possibilities are listed in tables. It turned out that only the Minkowski space has isometry group of maximal dimension 10, which is the Poincare group. Among all possibilities we are interested in those whose orbits are Σ_t. Thus the dimension of G is at least 3. There are three possibilities of 6-dimensional such isometry groups, which correspond to FRW spacetimes. A number of possibilities are available with 4-dimensional groups, which correspond to the LRS spacetimes. And finally there are 9 classes of 3-dimensional real Lie groups $Bi(N)$ (called Bianchi groups), which together with their factors $Bi(N)/\Gamma$ by discrete subgroups Γ represent the isometry groups of the spatially homogeneous spacetimes. It turned out further, that in all these cases besides one (the so called Kantowski-Sachs model) the isometry group is the semidirect product $G = \Sigma \rtimes O$. In this case we will call $\mathcal{T}_t$ a *semidirect homogeneous vector bundle.* In particular, for 6-dimensional FRW groups, 4-dimensional LRS groups and 3-dimensional Bianchi groups $O = SO(3)$, $SO(2)$ and $\{1\}$, respectively. The normal subgroups Σ are nothing else than $Bi(N)/\Gamma$.

Further in this chapter we will work on the semidirect homogeneous vector bundles. After establishing the necessary mathematical framework, we will obtain results concerning the structure of G-invariant homogeneous bi-distributions. In the next chapter we will concentrate on FRW and Bianchi models, which are of primary interest in cosmology. Harmonic and spectral analysis will be performed for concrete groups.

2.2 On harmonic analysis in semidirect homogeneous vector bundles

In this section we will collect information on harmonic analysis in G-homogeneous vector bundles $\mathcal{T} \to G/O$ where $G = M \rtimes O$ which will be useful later in the work. This does not pretend to be self-contained or systematic; quite the contrary, we will introduce mainly what we were not able to find in the literature. Otherwise references will be provided.

Semidirect homogeneous vector bundles. Let $G = M \rtimes O$, where O is a compact connected type I Lie subgroup, and M a connected normal type I Lie subgroup. Moreover, we demand that the modular function of M has a non-trivial kernel, so that the representation theories of both M and G are well under control by Theorem 7.50 of [21]. We note that this is the case for all Bianchi groups which are in fact the only candidates for M in our context. Let $M = G/O$ have a Riemannian structure h which is invariant under the left action of G. Let further $\mathcal{T} \to M$ be an n-dimensional (real or complex) vector bundle with standard fiber V and a pseudo-Riemannian fiber metric $\mathfrak{g}$. Let there be a smooth left action of G on $\mathcal{T}$ covering the left multiplication of G on the base, such that the fiber metric is invariant under that action. Then we will call $\mathcal{T}$ a *semidirect G-homogeneous vector bundle*. If we choose an orthonormal frame $\{X_i\}$ of T^*_1M (or $\{Y_i\}$ of $\mathcal{T}|_1$), and drag it throughout M using the transitive left action of G, we will obtain G-invariant global smooth frame $\{X_i\}$ in T^*M (similarly, $\{Y_i\}$ in $\mathcal{T}$). Thus both T^*M and $\mathcal{T}$ are trivial bundles. Associated to the Riemannian structure h there is a Laplace operator Δ acting on sections $f \in C^\infty(\mathcal{T})$.

The regular and quasi-regular representations for the line bundle. Suppose $\mathcal{T}$ from above is a line bundle, $n = 1$. The left regular representation L_g of G on $C(G)$ acts as

$$L_g f(x) = f(g^{-1}x),\ \forall g, x \in G.$$

Because the Riemannian structure is G-invariant, the metric measure dx is a left Haar measure on G, and hence L_g is a unitary representation on $L^2(G)$.

Now any point $x \in G$ can be uniquely written as $x = x_M x_O$, where $x_M \in M$ and $x_O \in O$. Let dx_M be the metric driven left G-invariant measure on M, and dx_O the Lebesgue measure on O normalized to $|O| = 1$. Then $dx = dx_M dx_O$ gives a left Haar measure on G. Functions f on M are identified with their right O-invariant extensions to G, i.e., $f(xo) = f(x) = f(xO)$, for any $x \in G$, $o \in O$. Thus $C(M) \in C(G)$ (similarly $L^2(M) \in L^2(G), etc.$) and we may consider the restriction U_g of the left regular representation L_g on $C(G)$ to $C(M)$. Its action will be given by

$$U_g f(x_M O) = f(g^{-1}x_M O),\ \forall x_M \in M,\ g \in G.$$

The representation U_g of G is the left quasi-regular representation, and it is nothing else but the induced representation $\mathrm{Ind}_O^G 1$. Note that for $O = \{1\}$ we simply have $G = M$ and $L_g = U_g$.

Neither L_g nor U_g need to be irreducible. The central decomposition of L_g is

$$L_g = \int_{\hat{G}}^{\oplus} d\nu(\pi) L_g(\pi),$$

where $\nu(\pi)$ is the Plancherel measure and $L_g(\pi) = \pi \otimes 1$ is the primary representation composed of $mult(\pi, L_g) = \dim \bar{\pi} \in [1, \infty]$ copies of π [21]. The central decomposition of U_g will be

$$U_g = \int_{\hat{G}_M}^{\oplus} d\mu(\pi) U_g(\pi),$$

where $\hat{G}_M \subset \hat{G}$, $d\mu$ is the spectral measure of U_g and for μ-almost all π, $U_g(\pi)$ is a multiple of π (multiplicities $mult(\pi, U_g)$ and the measure $d\mu(\pi)$ need to be determined). The corresponding Hilbert space decompositions are

$$L^2(G) = \int_{\hat{G}}^{\oplus} d\nu(\pi) \mathcal{H}_\pi \otimes \mathcal{H}_{\bar{\pi}}$$

and

$$L^2(M) = \int_{\hat{G}_M}^{\oplus} d\mu(\pi) \mathcal{H}(\pi),$$

where $\mathcal{H}(\pi) = \mathcal{H}_\pi \otimes \mathbb{C}^{mult(\pi,U_g)} \subset \mathcal{H}_\pi \otimes \mathcal{H}_{\bar{\pi}}$. Here $\mathbb{C}^{mult(\pi,U_g)}$ symbolizes some Hilbert space of dimension $mult(\pi, U_g)$ which is finite or infinite.

In the following we will deal with U_g keeping in mind that in case $G = M$ everything reduces to L_g.

The operator Π_π. Consider for any $\pi \in \hat{G}$ the bounded operator

$$\Pi_\pi = \int_O do\, \pi(o).$$

Then Π_π is self adjoint,

$$\Pi_\pi^* = \int_O do\, \pi(o)^* = \int_O do\, \pi(o^{-1}) = \Pi_\pi.$$

Moreover, because O is unimodular, we have

$$\pi(o)\Pi_\pi = \pi(o) \int_O do' \pi(o') = \int_O d(oo')\pi(oo') = \Pi_\pi = \Pi_\pi \pi(o),\ \forall o \in O,$$

and hence Π_π is a projection,

$$\Pi_\pi^2 = \int_O do\pi(o)\Pi_\pi = \int_O do\Pi_\pi = \Pi_\pi.$$

Π_π is a projection onto an invariant subspace of $\pi|_O$. Recall the operator D_π of [21] which satisfied $D_\pi\pi(x) = \Delta^{\frac{1}{2}}(x)\pi(x)D_\pi$, for all $x \in G$. In particular, we find that $D_\pi\pi(o) = \pi(o)D_\pi$ for all $o \in O$, and consequently, $D_\pi\Pi_\pi = \Pi_\pi D_\pi$.

The Fourier transform in G/O. The Fourier transform in $M = G/O$ associated to U_g is naturally the restriction of that on G associated to L_g; for μ-almost all $\pi \in \hat{G}_M$

$$\hat{f}(\pi) = \pi(f)D_\pi \in \mathcal{H}(\pi).$$

For any $f \in C_0(M)$ and μ-almost all $\pi \in \hat{G}_M$ we have

$$\pi(f) = \int_M dx_M \int_O dx_O f(x_M O)\pi(x_M)\pi(x_O) = \int_M dx_M f(x_M O)\pi(x_M)\Pi_\pi. \qquad (2.1)$$

As usual we have $\pi(U_g f) = \pi(L_g f) = \pi(g)\pi(f)$ for $g \in G$, $f \in C_0(M)$. The convolution $f * h$ has the property that if $f \in C_0(G)$ and $h \in C_0(M)$ then $f * h \in C_0(M)$. Moreover, it satisfies $\pi(f * h) = \pi(f)\pi(h)$.

The case of arbitrary $\mathcal{T}$. Let now $\dim V = n \geq 1$. The left quasi-regular representation of G on $C^\infty(\mathcal{T})$ acts as

$$U_g^{\mathcal{T}} f(x) = g^{-1} f(g^{-1}x), \forall f \in C^\infty(\mathcal{T}).$$

Recall the G-invariant orthonormal frame $\{Y_i\}_{i=1}^n$ in $\mathcal{T}$ and write any $f \in C^\infty(\mathcal{T})$ as $f = \sum f^i Y_i$. Using that $U_g^{\mathcal{T}} Y_i = Y_i$ we find

$$U_g^{\mathcal{T}} f(x) = \sum_{i=1}^n U_g f^i \times Y_i,$$

where U_g is the left quasi-regular representation of G on $C^\infty(M)$. Thus $U_g^{\mathcal{T}} = \oplus_n U_g$, and the harmonic analysis of $U_g^{\mathcal{T}}$ is the same as that of U_g except that each primary subrepresentation of $U_g^{\mathcal{T}}$ is the n-fold copy of the corresponding primary subrepresentation of U_g. Making the identification $C_0^\infty(\mathcal{T}) \ni f \to \{f^i\} \in \oplus_n C_0^\infty(M)$ we find the Fourier transform of $f \in C_0^\infty(\mathcal{T})$ to be

$$\hat{f}(\pi) = \oplus_{i=1}^n \hat{f}^i(\pi),$$

or to say in words, a matrix with n times more columns than that of a scalar function. The inverse Fourier transform will be

$$f(x) = \sum_{i=1}^{n} \int_{\hat{G}_M} d\mu(\pi) Tr \left[D_\pi \Pi_\pi \pi^*(x) \hat{f}^i(\pi) \right] \times Y_i(x).$$

2.3 On the Fourier transform of distributions

Here we will collect miscellaneous facts about distributions and their Fourier transform, which we did not meet in the literature. We continue working with the semidirect homogeneous vector bundle $\mathcal{T}$ with notations established earlier.

Let $\hat{\mathcal{D}}^{\mathcal{T}}(\hat{G}_M)$ be the image of $D(\mathcal{T}) = C_0^\infty(\mathcal{T})$ under the harmonic analytical Fourier transform $f(x_M) \to \hat{f}(\pi)$. As we have already seen, $\hat{f}(\pi) = \oplus_n \hat{f}^i(\pi)$, hence $\hat{\mathcal{D}}^{\mathcal{T}}(\hat{G}_M) = \oplus_n \hat{\mathcal{D}}(M)$, where $\hat{\mathcal{D}}(M)$ is the image under the Fourier transform of $C_0^\infty(M)$. $\hat{\mathcal{D}}^{\mathcal{T}}(\hat{G}_M)$ inherits the topology of $D(\mathcal{T})$ via the Fourier transform, and one can consider the Fourier transform of distributions $D(\mathcal{T})' \ni u \to \hat{u} \in \hat{\mathcal{D}}^{\mathcal{T}}(\hat{G}_M)'$ given by $\hat{u}(\hat{f}) = u(f)$.

The Fourier transform has the remarkable property that it interchanges the local and global behaviors. Namely, the local irregularities of a function f are reflected in the decay properties of $\hat{f}(\pi)$ at large π, and conversely, the behavior at infinity of f determines the local regularity of $\hat{f}(\pi)$. The precise description of these phenomena requires a thorough functional analytical investigation, which we, unfortunately, have no possibility to perform here.

It is widely known that any distribution restricted to a compact region is of finite order. In [23] the general structure of distributions of finite order has been found for $D(\mathbb{R}^n)$. Following a similar pattern we present here a partial generalization of that result. By **Proposition 5.1** let us choose the topology $(X_i, 2, l^2)$ for convenience.

Proposition 2.1 *Let $\mathcal{T}_K$ be an n-dimensional (complex) pseudo-Riemannian vector bundle over a connected parallelizable (pseudo-)Riemannian manifold K, and let ∇ be a fiber* *metric connection. Every $u \in D(\mathcal{T}_K)'$ of finite order has a representation*

$$u(f) = \sum_{q \leq k} (F_{\alpha,q}, P_{\alpha,q}(X_i)f)_2, \; \forall f \in D(\mathcal{T}),$$

where $F_{\alpha,q} \in L^2(\mathcal{T}_K)$ and the smallest possible such k is the order of u.

Proof: By our choice

$$\|f\|_k = \sqrt{\sum_{q \leq k} \|P_{\alpha,q}(X_i)f\|_2^2}.$$

Let k be the order of u, i.e., u is continuous in $\|.\|_k$-norm. Define the following linear injective map

$$\mathcal{V} : D(\mathcal{T}_K) \to \Phi = \bigoplus_{q \leq k} L^2(\mathcal{T}_K)$$

by

$$\mathcal{V}(f) = \bigoplus_{q \leq k} P_{\alpha,q}(X_i)f.$$

Then obviously $\|\mathcal{V}(f)\|_\Phi = \|f\|_k$. If we denote by $\Psi = \mathcal{V}\left(D(\mathcal{T}_K)\right) \subset \Phi$, then $u \circ \mathcal{V}^{-1}$ is a continuous functional on Ψ with the norm $\|.\|_\Phi$, and thus by Hahn-Banach theorem can be extended to a continuous functional $F \in \Phi'$. But Φ is a Hilbert space, thus $\Phi' = \Phi$ and $F \in \Phi$, and for any $\phi \in \Phi$,

$$F(\phi) = \sum_{q \leq k} (F_{\alpha,q}, \phi_{\alpha,q})_2, \; F_{\alpha,q} \in L^2(\mathcal{T}).$$

This yields our desired formula

$$u(f) = \sum_{q \leq k} (F_{\alpha,q}, P_{\alpha,q}(X_i)f)_2.$$

If such a formula held for a smaller k, then obviously the order of u would be smaller. □

Several variations of this proposition may be established by choosing different norms. Note that the order of a distribution, if finite, depends on the choice of the family of norms defining the topology.

Remark 2.1 *As already mentioned, any distribution is locally of finite order, hence the proposition applies to the restriction $u_K \in C_0^\infty(\mathcal{T}|_K)'$ of any $u \in D(\mathcal{T})'$ to arbitrary compact connected region $K \subset M$.*

We come back to our homogeneous bundle $\mathcal{T}$ and proceed to the Fourier description of distributions $u \in D(\mathcal{T})'$ of finite order, which again can be applied for restrictions to compact regions.

Proposition 2.2 *Any distribution $u \in D(\mathcal{T})'$ of finite order is given by*

$$u(f) = \int_{\hat{G}_M} d\mu(\pi) Tr \left[\hat{u}(\pi)^* \hat{f}(\pi) \right],$$

where $\hat{u}(\pi) : \mathbb{C}^{mult(\pi, U_g)*n} \to \mathcal{H}_\pi$ is a μ-locally integrable field of Hilbert-Schmidt operators. (Note that the trace operator includes also the summation by fiber indices $i = 1, .., n$, which now enumerate blocks of columns.)

Proof: Let k be the order of u. Choose $\{X_i\}$ to be the generators of left translations on $C^\infty(\mathcal{T})$ and let by **Proposition 2.1** write u as

$$u(f) = \sum_{q \le k} (F_{\alpha,q}, P_{\alpha,q}(X_i) f)_2.$$

Consider the Fourier transform

$$\widehat{X_i f}(\pi) = \int_M dx_M \left(\lim_{t \to 0} \frac{(U_{\exp(-t\xi_i)} - 1) f(x_M)}{t} \right) \pi(x_M) \Pi_\pi D_\pi$$

where ξ_i is the corresponding element of the Lie algebra of M. The integral runs over a compact region, and is therefore uniformly absolutely convergent with the Hilbert-Schmidt norm, thus we can interchange the limit with the integral,

$$\widehat{X_i f}(\pi) = \lim_{t \to 0} \frac{1}{t} \int_M dx_M (U_{\exp(-t\xi_i)} - 1) f(x_M) \pi(x_M) \Pi_\pi D_\pi = \lim_{t \to 0} \frac{\pi(\exp(-t\xi_i)) - 1}{t} \hat{f}(\pi).$$

On the right hand we see nothing else but the generator of the derived representation of π,

$$\lim_{t \to 0} \frac{\pi(\exp(-t\xi_i)) - 1}{t} = -\partial_i \pi,$$

whence we find

$$\widehat{X_i f}(\pi) = -\partial_i \pi \hat{f}(\pi).$$

As a result we have

$$\widehat{P_{\alpha,q}(X_i) f}(\pi) = P_{\alpha,q}(-\partial_i \pi) \hat{f}(\pi),$$

and thereby

$$u(f) = \sum_{q \le k} \int_{\hat{G}_M} d\mu(\pi) Tr \left[\hat{F}_{\alpha,q}(\pi)^* P_{\alpha,q}(-\partial_i \pi) \hat{f}(\pi) \right] = \int_{\hat{G}_M} d\mu(\pi) Tr \left[\hat{u}(\pi)^* \hat{f}(\pi) \right],$$

where

$$\hat{u}(\pi) = \sum_{q \le k} \left[P_{\alpha,q}(-\partial_i \pi) \right]^* \hat{F}_{\alpha,q}(\pi).$$

This completes the proof. □

Such a result should not be surprising. If the measurable functions $F_{\alpha,q}$ were q times differentiable within the space of locally integrable functions, then we could hypothetically use integration by parts to make all the terms in the formula of **Proposition 2.1** of order 0, which would correspond to a regular distribution. The failure of the derivatives of $F_{\alpha,q}$ to remain locally integrable is reflected in the fact, that multiplication of $\widehat{F}_{\alpha,q}(\pi)$ by $\partial_i \pi^*$ makes it not square integrable any more, but possibly only locally integrable. This reflects the local-to-global interchange made by the Fourier transform: higher frequencies feel local irregularities.

The image $\hat{\mathcal{D}}^{\mathcal{T}}(\hat{G}_M)$ of compactly supported smooth sections under the Fourier transform is of considerable interest. In harmonic analysis it is described by various Paley-Wiener type theorems. Although there are refined Paley-Wiener theorems for adapted Fourier transforms for certain classes of semisimple or solvable groups, there seems to be no such theory for the general abstract setup. Next we present a partial answer to the problem, namely, a criterion for smoothness for sufficiently decaying functions, which gives hints about how the general solution might look like.

Proposition 2.3 *For a function $f \in L^2(\mathcal{T})$ the following two conditions are equivalent:*

(i) for any polynomial $P(X_i)$ of generators $\{X_i\}$ with constant coefficients, $P(X_i) f \in L^2(\mathcal{T})$

(ii) $\hat{f}(\pi)$ decays at infinity of $\hat{G}_M$ faster than the inverse of any polynomial in the generators $\partial_i \pi^*$

Proof: As we have seen in the proof of the previous proposition,

$$\widehat{P(X_i)f} = P(-\partial_i\pi)^*\hat{f}(\pi),$$

and the requirement that $\widehat{P(X_i)f} \in L^2(\hat{G}_M)$ for any $P(X_i)$ is equivalent to the assertion (ii) of the proposition. $\square$

We can go a step further and establish a weaker necessary condition for a distribution to be given by a smooth integral kernel. For this purpose we want to remind a few definitions on a more abstract level.

Let $\mathcal{D}(S)$ be a test function space. We have $\mathcal{D}(S) \subset L^\infty(S)$ and therefore $L^\infty(S)' \subset \mathcal{D}(S)'$. Let $\{\eta_i\}$ be a finite system of linear maps $\eta_i : S \to S$. A distribution $u \in \mathcal{D}(S)'$ is of *rapid decay* in $\{\eta_i\}$ if for any polynomial $P(\eta_i)$ of variables $\{\eta_i\}$ it holds $u(P(\eta_i).) \in L^\infty(S)'$. We will symbolically write this as $u = \mathfrak{o}(\{\eta_i\}^{-\infty})$. If u is given by a locally integrable kernel, and $\{\eta_i\}$ are coordinate operators, then this definition obviously reduces to the usual criterion for functions of rapid decay.

Proposition 2.4 *For a distribution $u \in \mathcal{D}(\mathcal{T})'$ from $\hat{\mathcal{D}}^{\mathcal{T}}(\hat{G}_M)' \ni \hat{u} = \mathfrak{o}(\{\partial_i\pi\}^{-\infty})$ it follows that u has a smooth integral kernel.*

Proof: That u is smooth means that all derivatives of all fiber components u^j are continuous. In other words, for any polynomial P in the generators $\{X_i\}$, the distributions $P(X_i)u^j$ can be evaluated pointwise. A precise statement can be given as follows. u is smooth if and only if for any polynomial $P(X_i)$, point $m \in M$ and sequence of test functions $f_q \to \delta(x-m)$ in $C_0^\infty(M)'$, the following limit exists for all $j = 1, ..., n$ and is finite, $\lim_{q\to\infty} u^j(P(-X_i)f_q)$. The Fourier transform of the distribution $\delta_m = \delta(x-m)$ can be easily read from the Fourier inversion formula,

$$\hat{\delta}_m(\hat{f}) = \int_{\hat{G}_M} d\mu(\pi) Tr\left[D_\pi \Pi_\pi \pi^*(m)\hat{f}(\pi)\right].$$

That means $f_q \to \delta(x-m)$ is equivalent to $\hat{f}_q \to \pi(m)\Pi_\pi D_\pi$ in the weak sense. Hence

$$\widehat{P(-X_i)f_q} \to P(\partial_i\pi)\pi(m)\Pi_\pi D_\pi$$

in the weak topology. It follows

$$\lim_{q\to\infty} u^j(P(-X_i)f_q) = \hat{u}^j\left(P(\partial_i\pi)\pi(m)\Pi_\pi D_\pi\right) \tag{2.2}$$

whenever one of the sides converges.

Now suppose $\hat{u} = \mathfrak{o}(\{\partial_i\pi\}^{-\infty})$. Then for any $\hat{f} \in L^\infty(\hat{G}_M)$ (i.e., $\|\hat{f}(\pi)\| \in L^\infty(\hat{G}_M)$ in the usual sense) we have

$$\hat{u}^j\left(P(\partial_i\pi)\hat{f}^j(\pi)\right) < \infty,\ j = 1, ..., n.$$

In particular, $\pi(m)\Pi_\pi D_\pi \in L^\infty(\hat{G}_M)$, whence (Eq.2.2) follows. □

We are incline to think that this necessary condition is not far from the desirable equivalent condition. This is, however, an open problem in harmonic analysis, and we only hope to be able to give a satisfactory answer in the future at least in the context we are interested in.

2.4 The adapted Fourier transform

We start by noting that because the function $m^\star(t)$ is a function of time only, the eigenfunctions of D_{Σ_t} are nothing else but the eigenfunctions of the Laplace operator Δ_t. In the first chapter we introduced the eigenfunction decomposition associated to any self adjoint operator as the Laplace operator Δ,

$$f \to \tilde{f}(\alpha) = \zeta_\alpha(f),$$

where ζ_α-s are the generalized eigenfunctions of Δ. Putting additional structure related with particular geometries one arrives at various Fourier transforms, which are very practical in many respects. On the other hand, the abstract harmonic analytical Fourier transform is a powerful tool for analyzing general problems and properties, but its machinery is functional analytically complicated for use. These two theories are, however, related, although the exact relations have not been sufficiently explored in the literature so far except for compact groups. In the compact case the eigenfunctions of Δ are the matrix elements of the irreducible representations for some choice of the basis, and the two techniques can be unified. Each choice of the basis results in a Fourier transform which is adapted to it, hence such transforms are sometimes called adapted Fourier transforms. In the non-compact case functional analytical complications arise, though intuitively the situation remains similar. In this section we will try to construct adapted Fourier transforms at least on our semidirect homogeneous bundle $\mathcal{T}$.

The Laplace operator Δ is invariant under G and hence commutes with $U_g^{\mathcal{T}}$. This means on each primary component it acts as a multiplication from the right by a possibly unbounded self-adjoint operator $\hat{\Delta}(\pi)$,

$$\widehat{\Delta f}(\pi) = \hat{f}(\pi)\hat{\Delta}(\pi).$$

For any $f \in L^2(\mathcal{T})$ we have that Δf is a distribution of order at most 2. By **Proposition 2.2** it means that the multiplication of any Hilbert-Schmidt operator $\hat{f}(\pi)$ by $\hat{\Delta}(\pi)$ from the right leaves it again Hilbert-Schmidt. Let $\sigma(\pi) \subset \mathbb{R}$ be the spectrum of the self-adjoint operator $\hat{\Delta}(\pi)$ as acting from the right (this spectrum is non-positive, because Δ is an elliptic operator). For each $\lambda \in \sigma(\pi)$ let $\hat{\xi}_{\pi,\lambda,r,s}$ be the generalized eigenfunctions of $\hat{\Delta}(\pi)$, i.e., distributions satisfying $\hat{\xi}_{\pi,\lambda,r,s}\hat{\Delta}(\pi) = \lambda\hat{\xi}_{\pi,\lambda,r,s}$ which are linearly independent and complete in $\mathcal{H}(\pi)$ for $r \in R_\pi \subset \mathbb{R}$ and $s \in S^n_{\pi,\lambda} \subset \mathbb{R}$ (they can be constructed from delta functions using the spectral theorem). Now consider the following distributions in the Fourier space,

$$\hat{\zeta}_{\pi,\lambda,r,s}(\pi') = \delta(\pi - \pi')\hat{\xi}_{\pi,\lambda,r,s}.$$

Their preimages are distributions $\zeta_{\pi,\lambda,r,s} \in D(\mathcal{T})'$ which are generalized eigenfunctions of Δ, and by elliptic regularity theorem, are smooth sections in $\mathcal{T}$. Thus we have found, that the adapted Fourier transform $\tilde{f}(\pi, \lambda, r, s)$ is nothing else but the coefficients of $\hat{f}(\pi)$ as expended in the system $\hat{\xi}_{\pi,\lambda,r,s}$. It is worth noting that r parameterizes $\mathcal{H}_\pi$, and λ, s parameterize $\mathbb{C}^{mult(\pi,U_g)} * n$. Actually, $S^n_{\pi,\lambda}$ consists of n copies of some set $S_{\pi,\lambda}$.

The choice of the system $\hat{\xi}_{\pi,\lambda,r,s}$ is rather arbitrary and leaves room for adaptations. The first adaptation we wish to make is the following. For any $\zeta_{\pi,\lambda,r,s}$ we want $\bar{\zeta}_{\pi,\lambda,r,s} = \zeta_{\pi',\lambda',r',s'}$ for some other parameters. Obviously $\lambda = \lambda'$, and it is easy to see from the Fourier inversion formula, that this amounts to requiring that $\bar{\xi}_{\pi,\lambda,r,s}$ enters the system $\xi_{\bar{\pi},\lambda,r',s'}$ for the representation $\bar{\pi}$ with some other parameters r', s'. The representation $\bar{\pi}$ may lie in the same equivalence class $[\pi]$ or not.

Lie groups are analytic manifolds, and all the group and algebra structure is given by analytic functions in any analytic atlas. In particular, the eigenfunction problem $\Delta\zeta_{\pi,\lambda,r,s} = \lambda\zeta_{\pi,\lambda,r,s}$ is an analytic elliptic equation, and the solutions $\zeta_{\pi,\lambda,r,s}(x)$ are therefore analytic functions in x. If M is compact, then $\hat{G}_M$ is discrete, and each $\sigma(\pi)$ is also discrete. Representations are finite dimensional, hence r and s run over finite sets. The set $\tilde{\Sigma} = \{\alpha = (\pi, \lambda, r, s)\}$ can be considered a discrete manifold symbolically divided into n components as corresponding to each copy of $S_{\pi,\lambda}$. The space $\hat{\mathcal{D}}^{\mathcal{T}}(\hat{G}_M)$ corresponds now to the space $\tilde{D}(\tilde{\Sigma})$ of functions on $\tilde{\Sigma}$, which are of rapid decay in λ, and also decay sufficiently fast in π by **Proposition 2.3**.

If M is non-compact, suppose there exists a subset $\tilde{K} \subset \hat{G}_M$ such that $\mu(\hat{G}_M \setminus \tilde{K}) = 0$ and $\tilde{K}$ can be cast into an analytic manifold. Then we can restrict our Fourier transform from $\hat{G}_M$ to $\tilde{K}$ without violation of the Plancherel equality. Suppose further that the set $\tilde{\Sigma} = \{\alpha = (\pi, \lambda, r, s)\}$ can be made an analytic manifold consisting of n disjoint components as in the compact case. Each component itself may have several connected components if $1 < mult(\pi, U_g) < \infty$, in which case s will run over a discrete set. Then we can choose $\zeta_{\pi,\lambda,r,s}$ to be analytic in all its parameters (if s is discrete, analyticity in s is void), so that $\hat{\mathcal{D}}^{\mathcal{T}}(\hat{G}_M)$ will correspond to the space $\tilde{D}(\tilde{\Sigma})$ of some analytic functions on $\tilde{\Sigma}$ which have at least above mentioned decay properties in λ and π, but also are L^2 in r, and in s if the latter is continuous.

Finally let us define a symbolic involution $\alpha \to -\alpha$ on $\tilde{\Sigma}$ satisfying $\zeta_{-\alpha} = \bar{\zeta}_\alpha$. Clearly this involution will preserve λ. Now if the necessary assumptions are satisfied, we arrive at a conventional Fourier transform. Later we will see that in the majority of situations in cosmology these assumptions are valid, and that will enable us to exploit the machinery of mode decomposition to our cosmological models.

2.5 Invariant bi-distributions

In this section we will try to analyze the structure of bi-distributions $w \in (\mathcal{D}(\mathcal{T}) \otimes \mathcal{D}(\mathcal{T}))'$ which are invariant under the left quasi-regular action $U_g^{\mathcal{T}}$ of G on $\mathcal{D}(\mathcal{T})$,

$$w(U_g^{\mathcal{T}} f, U_g^{\mathcal{T}} h) = w(f, h), \forall f, h \in \mathcal{D}(\mathcal{T}),$$

and compare with results obtained earlier in the literature.

Decomposing each $f = \sum f^i Y_i$, $f^i \in C_0^\infty(M)$, we find for $u \in \mathcal{D}(\mathcal{T})'$ and $w \in (\mathcal{D}(\mathcal{T}) \otimes \mathcal{D}(\mathcal{T}))'$

$$u(f) = \sum_{i=1}^{n} u^i(f^i),\ w(f,h) = \sum_{i,j=1}^{n} w^{ij}(f^i, h^j),\ u^i \in C_0^\infty(M)',\ w^{ij} \in (C_0^\infty(M) \otimes C_0^\infty(M))',$$

so that the problem reduces to that for scalar distributions.

The following proposition establishes the general form of the G-invariant (or homogeneous) bi-distributions. Our approach is greatly inspired by [43] where this analysis is performed for $\mathbb{R}^n$.

Proposition 2.5 *Every $w \in (C_0^\infty(M) \otimes C_0^\infty(M))'$ satisfying $w(U_g f, U_g h) = w(f,h)$, $\forall f, h \in C_0^\infty(M)$, $g \in G$, has the form*

$$w(f,h) = u_w\left(\bar{f}^* * h\right)$$

for some $u_w \in C_0^\infty(M)'$. And conversely, any $u_w \in C_0^\infty(M)'$ gives rise to such an invariant bi-distribution w.

Proof: Recall that for scalar functions $U_g f(x_M O) = f(g^{-1} x_M O)$. By the nuclear theorem w can be uniquely extended to $\tilde{w} \in C_0^\infty(M \times M)'$ via embedding

$$C_0^\infty(M) \otimes C_0^\infty(M) \ni f(x_M) \otimes h(y_M) \to f(x_M)h(y_M) \in C_0^\infty(G \times G).$$

That

$$w(f(g^{-1}x_M O), h(g^{-1}y_M O)) = w(f,h)$$

by continuity implies that

$$\tilde{w}(\phi(g^{-1}x_M O, g^{-1}y_M O)) = \tilde{w}(\phi(x_M, y_M)),\ \forall \phi \in C_0^\infty(M \times M).$$

Define the linear automorphism

$$C_0^\infty(M \times M) \ni \phi(x_M, y_M) \to \psi_\phi(x_M, y_M) \in C_0^\infty(M \times M)$$

by

$$\psi_\phi(x_M, y_M) = \int_O dx_O \phi(x_M, x_M x_O y_M O) = \int_O dx_O \phi(x_M, x_M O x_O y_M O),$$

and the pullback of $\tilde{w}$ under this automorphism by $\tilde{v}$, $\tilde{v}(\psi_\phi) = \tilde{w}(\phi)$. If $\phi_g(x_M, y_M) =$

$\phi(g^{-1}x_MO, g^{-1}y_MO)$ then

$$\psi_{\phi_g}(x_M, y_M) = \int_O dx_O \phi(g^{-1}x_MO, g^{-1}x_MOx_Oy_MO) = \psi_\phi(g^{-1}x_MO, y_M).$$

Now

$$\tilde{w}(\phi_g) = \tilde{v}(\psi_{\phi_g}) = \tilde{v}(\psi_\phi) = \tilde{w}(\phi),$$

thus

$$\tilde{v}(\psi_\phi(x_M, y_M)) = \tilde{v}(\psi_\phi(g^{-1}x_MO, y_M)), \forall g \in G.$$

Consider the restriction v of $\tilde{v}$ to $C_0^\infty(M) \otimes C_0^\infty(M)$. The last equation implies $v(f(g^{-1}x_MO), h(y_M)) = v(f, h)$, $\forall f, h \in C_0^\infty(M)$. If we fix h, then $v(., h) \in C_0^\infty(M)'$ is a distribution which is invariant under all translations, and is thus given by a constant kernel, $v(f, h) = u_w(h) \int_M dx_M f(x_M)$, for some $u_w : C_0^\infty(M) \to \mathbb{C}$. On the other hand, if we fix f, then continuity in h implies $u_w \in C_0^\infty(M)'$. Because the integral $\int_M dx_M f(x_M)$ runs over a compact region, it can be transferred into u_w, i.e., $v(f, h) = u_w \left(\int_M dx_M f(x_M) h(y_M) \right)$. This in turn implies by continuity, that $\tilde{v}(\psi(x_M, y_M)) = u_w \left(\int_M dx_M \psi(x_M, y_M) \right)$. Finally we arrive at

$$w(f, h) = \tilde{w}(f(x_M)h(y_M)) = \tilde{v}(f(x_M) \int_O dx_O h(x_M x_o y_M O)) =$$

$$= u_w \left(\int_M dx_M f(x_M) \int_O dx_O h(x_M x_O y_M O) \right) = u_w(\bar{f}^* * h).$$

The converse statement is obvious. □

For a distribution $w \in (\mathcal{D}(\mathcal{T}) \otimes \mathcal{D}(\mathcal{T}))'$ this will mean

$$w(f, h) = \sum_{i,j=1}^n u_w^{ij} \left((\bar{f}^i)^* * h^j \right).$$

Remark 2.2 *Note that any G-invariant bi-distributions $w \in (C_0^\infty(M) \otimes C_0^\infty(M))'$ is in particular M-invariant. Let $f * h$ ($f \star h$) and f^* ($f^\star$) denote the convolution and the involution with respect to G (M), respectively. Then*

$$w(f, h) = u_w \left(\int_M dx_M f(x_M) \int_O dx_O h(x_M x_O y_M O) \right) =$$

$$= u_w \left(\int_O dx_O L_{x_O^{-1}} \bar{f}^\star \star h(y_M O) \right) = u'_w(\bar{f}^\star \star h)$$

for some other $u'_w \in (C_0^\infty(M) \otimes C_0^\infty(M))'$ *as expected.*

Let $\hat{\mathcal{D}}(\hat{G}_M)$ be the image of $C_0^\infty(M)$ under the harmonic analytical Fourier transform $f(x_M) \to \hat{f}(\pi)$. As an obvious corollary we arrive at the form of an invariant bi-distribution in the Fourier space.

Corollary 2.1 *A G-invariant bi-distribution* $w \in (C_0^\infty(M) \otimes C_0^\infty(M))'$ *in the Fourier space is given by*

$$w(f,h) = \hat{u}_w(\pi(\bar{f}^*)\hat{h}(\pi)) = \hat{u}_w(\pi(\bar{f})^*\hat{h}(\pi))$$

for some $\hat{u}_w \in \hat{\mathcal{D}}(\hat{G}_M)'$.

An immediate consequence of **Proposition 2.2** is the following

Corollary 2.2 *Under the assumptions of* **Proposition 2.2**, *a G-invariant bi-distribution* $w_K \in (C_0^\infty(K) \otimes C_0^\infty(K))'$ *with* $K \subset M$ *compact is given by*

$$w_K(f,h) = \int_{\hat{G}_M} d\mu(\pi) Tr\left[\hat{u}_K(\pi)^*\pi(\bar{f})^*\hat{h}(\pi)\right].$$

Proof: It suffices to note that

$$\text{supp}\{f * h\} \subset O(\text{supp}\{f\})^{-1}\text{supp}\{h\}O,$$

and to apply **Proposition 2.2**.

Finally we establish a generalization of the results by [32] for FRW spacetimes.

Proposition 2.6 *Suppose that the group G is such that all multiplicities* $mult(\pi, U_g)$ *are finite. Then any G-invariant bi-distribution* $w \in (D(\mathcal{T}) \otimes D(\mathcal{T}))'$ *has the form*

$$w(f,h) = \int_{\hat{G}_M} d\mu(\pi) Tr\left[(\hat{\bar{f}}(\pi)\hat{u}(\pi))^*\hat{h}(\pi)\right],$$

where $\hat{u}(\pi)$ *is a* μ*-locally measurable field of* $[mult(\pi, U_g) \cdot n] \times [mult(\pi, U_g) \cdot n]$ *complex matrices.*

Proof: Let start with the case $w \in (C_0^\infty(M) \otimes C_0^\infty(M))'$. The condition that the modular function of M has a nontrivial kernel ensures that the formula (7.49) of (Folland) is valid, so that for μ-almost all π the operator D_π is invertible (injective). Therefore we can write $\pi(f) = \hat{f}(\pi)D_\pi^{-1}$, so that $\pi(\bar{f})^*\hat{h}(\pi) = D_\pi^{-1}\hat{\bar{f}}(\pi)^*\hat{h}(\pi)$ where $\hat{\bar{f}}(\pi)^*\hat{h}(\pi)$ is a $mult(\pi, U_g) \times mult(\pi, U_g)$ complex matrix. Now for any compact $K \subset M$ by **Corollary 2.2** we find that the restriction w_K of w to $C_0^\infty(K) \otimes C_0^\infty(K)$ is given by

$$w_K(f,h) = \int_{\hat{G}_M} d\mu(\pi) Tr\left[\hat{u}'_K(\pi)^* D_\pi^{-1}\hat{\bar{f}}(\pi)^*\hat{h}(\pi)\right] = \int_{\hat{G}_M} d\mu(\pi) Tr\left[\hat{u}_K(\pi)^*\hat{\bar{f}}(\pi)^*\hat{h}(\pi)\right],$$

where $\hat{u}_K(\pi)$ is a $mult(\pi, U_g) \times mult(\pi, U_g)$ complex matrix. Choosing a larger compact $K \subset K' \subset M$ we will arrive at another matrix $\hat{u}_{K'}(\pi)$. But when restricted to K, $w_{K'}$ must coincide with w_K, hence $\hat{u}_{K'}(\pi) = \hat{u}_K(\pi)$. Thus the matrix $\hat{u}_K(\pi)$ is the same for any K, and the formula holds for the entire w.

Now for $w \in (D(\mathcal{T}) \otimes D(\mathcal{T}))'$ we have

$$w(f,h) = \int_{\hat{G}_M} d\mu(\pi) \sum_{i,j=1}^{n} Tr\left[(\hat{\bar{f}}^i(\pi)\hat{u}^{ij}(\pi))^*\hat{h}^j(\pi)\right] = \int_{\hat{G}_M} d\mu(\pi) Tr\left[(\hat{\bar{f}}(\pi)\hat{u}(\pi))^*\hat{h}(\pi)\right],$$

which completes the proof. □

In the case of FRW spacetimes all the assumptions of the last proposition are satisfied. In particular all $mult(\pi, U_g) = 1$ and for the scalar case we find that any G-invariant bi-distribution is given by a locally measurable scalar field $\hat{u}(\pi)$. The additional constraint of [32] of being polynomially bounded is an artifact of the imposed additional continuity requirement.

Chapter 3

Homogeneous and FRW cosmological spacetimes

3.1 The aims of the chapter

In this chapter we will not be as holistic as in previous two. Suppose we are dealing with a semidirect homogeneous space G/H. Given the harmonic analysis on G/H (the dual space, the spectral measure etc.) the general methods of the previous chapter allow us to apply the machinery of the harmonic analytical Fourier transform. On the other hand, given the (pseudo-)Riemannian structure and the spectral analysis on G/H (the metric and the connection, the spectral decomposition of Δ etc.) we can hopefully arrange a conventional Fourier transform and enjoy the merits of the first chapter. But unfortunately the harmonic and spectral analysis are done rather individually for different spaces, so that one needs to consider them case by case. For FRW spaces this is done since decades, and here we will merely present the most important information for our purposes.

The situation is different for Bianchi type spaces. The Bianchi I group is simply the additive group $\mathbb{R}^3$, of which the harmonic theory is well understood. The Bianchi II group is the so called Heisenberg group, which is also well known and its harmonic analysis is given in [56],[21],[54]. The Bianchi III group is isomorphic to the $ax + b$ group, of which the harmonic analysis is given in [21]. The Bianchi VIII group is the universal covering group of $SL(2,\mathbb{R})$, which is analyzed in [47]. The Bianchi IX group is the group $SU(2)$, which is extensively studied in the literature. However, little is known about groups IV-VII of Bianchi beyond the structure of their Lie algebras. To fill in this apparent gap we will give a unified harmonic analysis on all solvable Bianchi groups I-VII. This will be done based on the structure of their Lie algebras being semidirect products of $\mathbb{R}^2$ with $\mathbb{R}$, using the Mackey machine [21],[34]. In particular, we will obtain Plancherel measures for them explicitly.

The spectral theory of Δ is even more specific, as it depends on the choice of the metric and the connection. The eigenfunction problem of Δ is a 3-dimensional elliptic equation on a manifold without boundaries, which is not easy to solve even numerically. If we succeed to reduce it at least to a system of 1-dimensional equations, then we will say that the solutions can be found explicitly, meaning that they are given by special functions of one variable. For fiber dimension $n > 1$ this is a hard task. One step that can be done is to reduce this vector valued elliptic equation to a scalar elliptic equation with constraints on the holonomy subbundle of the principal bundle. But for non-trivial connections the holonomy bundle has a complicated geometry, and the problem is again very hard. This is why we do not go into details here. On the other hand the case of a line bundle (scalar field) is much easier. Then the only complication arises because of the arbitrarily chosen left invariant Riemannian metric. For FRW spaces this metric is given uniquely up to a scale factor. But for Bianchi type spaces the possibilities are rather large. We will perform a unified spectral decomposition on Bianchi groups II-VII, and construct the corresponding conventional Fourier transforms. For Bianchi VIII and IX we will present what is known in the literature.

Finally we will consider the question of mode decomposition of the first chapter on FRW and Bianchi spacetimes.

3.2 Semidirect structure of Bianchi I-VII groups

As a first step in the harmonic analysis on Bianchi groups we will try to explicitly realize the solvable Bianchi groups II-VII (I is Abelian and will serve as a starting point in the analysis of others) as semidirect products of Abelian subgroups. A classification of

I	II	III	IV	V	VI	VIII
0	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -q \end{pmatrix}$ $-1 < q \leq 1$	$\begin{pmatrix} p & -1 \\ 1 & p \end{pmatrix}$ $p \geq 0$

Table 3.1: The matrices M for Bianchi I-VII groups

solvable real Lie algebras with respect to such products can be inferred from [41].

Semidirect products of Lie algebras and Lie groups. We start by recalling some definitions. Let $\mathfrak{a}$ and $\mathfrak{b}$ be Lie algebras, and let $D(\mathfrak{a})$ be the Lie algebra of derivations on $\mathfrak{a}$. Let further $f : \mathfrak{b} \to D(\mathfrak{a})$ be a Lie algebra homomorphism. The *semidirect product Lie algebra* $\mathfrak{a} \times_f \mathfrak{b}$ is the algebra modeled on $\mathfrak{a} \oplus \mathfrak{b}$ with the Lie bracket

$$[(a,b),(a',b')] = ([a,a'] + f(b)a' - f(b')a, [b,b']),\ (a,b),(a',b') \in \mathfrak{a} \oplus \mathfrak{b}.$$

Let, on the other hand, A and B be Lie groups, and $F : B \to Aut(A)$ a Lie group homomorphism ($Aut(A)$ embedded into $GL(A)$). The *semidirect product* $A \times_F B$ of groups A and B is defined as the Lie group modeled on the product manifold $A \times B$ with the multiplication

$$(a,b)(a',b') = (aF(b)a', bb'),\ (a,b),(a',b') \in A \times B.$$

Following the notations of [31], denote by $F^\circ : B \to Aut(\mathfrak{a})$ the map $B \ni b \to d[F(b)] \in Aut(\mathfrak{a})$, where $\mathfrak{a}$ is the Lie algebra of A. Then the derivative of this map, $f = dF^\circ$, will be a Lie algebra homomorphism $f : \mathfrak{b} \to D(\mathfrak{a})$ ($\mathfrak{b}$ the Lie algebra of B), and the Lie algebra of the direct product Lie group $A \times_F B$ is the direct product Lie algebra $\mathfrak{a} \times_f \mathfrak{b}$ [31].

Bianchi I-VII groups as semidirect products. With this in mind, let us start with realizing Bianchi algebras I-VII as semidirect product algebras $\mathbb{R}^2 \times_f \mathbb{R}$ with some Lie algebra homomorphism $f : \mathbb{R} \to D(\mathbb{R}^2)$. This correspondence between Bianchi algebras and homomorphisms f can be obtained by combination of [33] and [41]. (Those uncomfortable with Russian may simply perform the semidirect construction and check the commutation relations.) Namely, in each case $f(r) = r \cdot M$, $r \in \mathbb{R}$, in a suitable basis, where M is a 2×2 matrix. The matrix M for each algebra is given in the Table 3.2.

The corresponding integral homomorphisms F° will be the exponentials $F^\circ(r) = e^{rM}$ (note that the exponential map on the group $\mathbb{R}$ is given by the identity map). If a

I	II	III	IV	V	VI	VIII
1	$\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$	$\begin{pmatrix} e^r & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} e^r & 0 \\ re^r & e^r \end{pmatrix}$	$\begin{pmatrix} e^r & 0 \\ 0 & e^r \end{pmatrix}$	$\begin{pmatrix} e^r & 0 \\ 0 & e^{-qr} \end{pmatrix}$ $-1 < q \leq 1$	$\begin{pmatrix} e^{pr}\cos(r) & -e^{pr}\sin(r) \\ e^{pr}\sin(r) & e^{pr}\cos(r) \end{pmatrix}$ $p \geq 0$

Table 3.2: The matrices $F(r)$ for Bianchi I-VII groups

diffeomorphism is given locally by a linear coordinate map, $x'_i = A_i^j x_j$ with the matrix A, then its differential will be given by the same matrix A. Now that $F^\circ(r) = d[F(r)]$ and that $F(r)$ are linear automorphisms, it follows that $F(r) = e^{rM}$. Thus all Bianchi groups I-VII are given by semidirect products $\mathbb{R}^2 \times_F \mathbb{R}$, where for each class the group homomorphism $F : \mathbb{R} \to Aut(\mathbb{R}^2)$ is given as in the Table 3.2.

It follows that the group multiplication is

$$(X, Y, X)(X', Y', Z') = ((X, Y) + F(Z)(X', Y'), Z + Z'),\ (X, Y, Z), (X', Y', Z') \in \mathbb{R}^2 \times_F \mathbb{R}.$$

The exponential map. Finally we note that all 7 groups are exponential, and the exponential map is given as follows. Let $(X, Y, Z) \in \mathbb{R}^2 \times_f \mathbb{R}$ with $(X, Y) \in \mathbb{R}^2$ and $Z \in \mathbb{R}$. We use the Zassenhaus formula

$$\exp(A + B) = \exp(A)\exp(B)\exp(C_2)\exp(C_3)...,$$

where coefficients C_m are homogeneous Lie elements composed of nested commutators of order m. We will use the convenient method of obtaining C_m recursively as given in [35]. If we set $A = (X, Y, 0)$ and $B = (0, 0, Z)$, we obtain

$$[A, B] = -f(Z)A.$$

Now equating the homogeneous summands of any order of (4.7) and (4.8) of [35], we obtain recursion formulas for C_m which are bulky in general. However, trying an ansatz $C_m = \alpha_m(-f)^{m-1}(Z)A$, $\alpha_m \in \mathbb{R}$, and checking directly for $m = 2$, one can easily prove it inductively, and find

$$\alpha_m = \frac{1 - m}{m!}.$$

It remains to calculate

$$\exp(C_2)\exp(C_3)... = \exp\left(\sum_{m=1}^{\infty} \frac{1 - m}{m!}(-f)^{m-1}(Z)A\right).$$

If $f(Z)$ is invertible for all Z then we write

$$\frac{1-m}{m!}(-f)^{m-1}(Z) = (-f)^{-1}(Z)\frac{(-f)^m(Z)}{m!} - \frac{(-f)^{m-1}(Z)}{(m-1)!},$$

and obtain

$$D(Z) \doteq \sum_{m=1}^{\infty} \frac{1-m}{m!}(-f)^{m-1}(Z) = (-f)^{-1}(Z)\left(e^{-f(Z)} - 1\right) - e^{-f(Z)} =$$

$$= f^{-1}(Z)\,(1 - F(-Z)) - F(-Z).$$

It is only for Bianchi II and III that $f(Z)$ is degenerate, and for these two we can compute directly

$$D(Z) = \sum_{m=1}^{\infty} \frac{1-m}{m!}(-f)^{m-1}(Z) = \frac{1}{2}f(Z) \text{ for Bianchi II}$$

and

$$D(Z) = \sum_{m=1}^{\infty} \frac{1-m}{m!}(-f)^{m-1}(Z) = (1-2e^{-1})f(Z) \text{ for Bianchi III.}$$

Thus we arrive at

$$\exp((X,Y,0) + (0,0,Z)) = \exp((X,Y,0))\exp((0,0,Z))\exp(D(Z)(X,Y),0).$$

The exponential maps of $\mathbb{R}^2$ and $\mathbb{R}$ are the identity maps, therefore

$$\exp((X,Y,Z)) = (X,Y,Z)(D(Z)(X,Y),0) = ([1 + F(Z)D(Z)](X,Y),Z),$$

where $F(Z)$ should be understood as $F(\exp(Z))$. The matrices $D(Z)$ appear somewhat bulky so we refrain from presenting them in a table.

The adjoint representations Ad **and** ad**.** Let $(g_x, g_y, g_z), (X,Y,Z) \in G$. The conjugation map $G \ni (X', Y', Z') = (g_x, g_y, g_z)(X,Y,Z)(g_x, g_y, g_z)^{-1}$ is given by

$$(X', Y', Z') = ((1 - F(Z))(g_x, g_y) + F(g_z)(X,Y), Z).$$

The adjoint representation Ad is the differential of this map at the identity $(X,Y,Z) =$

$(0,0,0)$, and so it is given by the matrix field Ad_g,

$$Ad_g = \begin{pmatrix} F(g_z) & & -F'(0)(g_x,g_y)^\top \\ & & \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly, using the equality $ad_X Y = [X,Y]$, the adjoint representation of the Lie algebra is given by the matrix

$$ad_{(X,Y,Z)} = \begin{pmatrix} f(Z) & & -f'(0)(X,Y)^\top \\ & & \\ 0 & 0 & 0 \end{pmatrix}.$$

The Haar measure and the modular function. The Haar measure on the Lie group is given by

$$d(exp(X,Y,Z)) = j(X,Y,Z)dXdYdZ,$$

where

$$j(X,Y,Z) = \mathfrak{h} \det \frac{1-e^{-ad_{(X,Y,Z)}}}{ad_{(X,Y,Z)}},$$

and $0 < \mathfrak{h} \in \mathbb{R}$ is an arbitrary constant. In group coordinates one can check that it is given by

$$dg = \mathfrak{h} \det F(-g_z) dg_z dg_y dg_z.$$

The groups are all non-compact, so there is no preferred normalization for the constant $\mathfrak{h}$. Later it will be determined as related to the chosen left invariant Riemannian metric on G. The modular function $\Delta(g) = \det Ad_g^{-1}$ can be readily seen to be $\Delta(g) = (\det F(-g_z))$.

This temporarily completes our task of analyzing the Bianchi I-VII groups as semidirect products. In the next section we will concentrate on their dual spaces.

3.3 The irreducible representations of Bianchi I-VII groups

In this section we will try to find the dual spaces of Bianchi I-VII groups using the Mackey procedure. Let us start with Bianchi I, which is simply the additive group $\mathbb{R}^3$. Its dual group $\hat{\mathbb{R}}^3$ is homeomorphic to itself, $\hat{\mathbb{R}}^3 = \mathbb{R}^3$, and the irreducible 1-dimensional representations are given by

$$\xi_{\vec{k}}(\vec{x}) = e^{i(\vec{k},\vec{x})},\ \vec{x} \in \mathbb{R}^3,\ \vec{k} \in \hat{\mathbb{R}}^3.$$

These scalar functions $\xi_{\vec{k}}$ can be viewed as unitary operator valued functions acting on the one complex dimensional Hilbert space $\mathbb{C}$.

The Mackey procedure for normal Abelian subgroups. We cite here the setup of the Mackey theory for groups with a normal Abelian subgroup as given in [21]. Let G be a locally compact group and N an Abelian normal subgroup. Then G acts on N by conjugation, and this induces an action of G on the dual group $\hat{N}$ defined by

$$g\nu(n) = \nu(g^{-1}ng),\ g \in G,\ \nu \in \hat{N},\ n \in N.$$

For each $\nu \in \hat{N}$, we denote by G_ν the stabilizer of ν,

$$G_\nu = \{g \in G\colon\ g\nu = \nu\},$$

which is a closed subgroup of G, and we denote by $\mathcal{O}_\nu$ the orbit of ν:

$$\mathcal{O}_\nu = \{g\nu\colon\ g \in G\}.$$

The action of G on $\hat{N}$ is said to be *regular* if some conditions are satisfied. To avoid presenting excessive information we only mention that if G is second countable (which is true for a Lie group), then the condition for a regular action is equivalent to the following: for each $\nu \in \hat{N}$, the natural map $gG_\nu \to g\nu$ from G/G_ν to $\mathcal{O}_\nu$ is a homeomorphism. In our case $\hat{N}$ is a smooth manifold, and the group actions are all smooth, hence this map is not only a homeomorphism but even a diffeomorphism. The constructions become simpler under the assumption that G is a semidirect product of N and the factor group $H = G/N$. We define the *little group* H_ν of $\nu \in \hat{N}$ to be $H_\nu = G_\nu \cap H$. Now we cite a beautiful theorem which appears as Theorem 6.42 in [21] and expresses the essence of the Mackey procedure. The functor **Ind** is defined on page 153 in the same monograph.

Theorem 3.1 (citation) *Suppose $G = N \ltimes H$, where N is Abelian and G acts regularly on $\hat{N}$. If $\nu \in \hat{N}$ and ρ is an irreducible representation of H_ν, then $\mathbf{Ind}_{G_\nu}^{G}(\nu\rho)$ is an irreducible representation of G, and every irreducible representation of G is equivalent to one of this form. Moreover, $\mathbf{Ind}_{G_\nu}^{G}(\nu\rho)$ and $\mathbf{Ind}_{G_\nu}^{G}(\nu'\rho')$ are equivalent if and only if ν and ν' belong to the same orbit, say $\nu' = g\nu$, and $h \to \rho(h)$ and $h \to \rho'(g^{-1}hg)$ are equivalent representations of H_ν.*

Application to the Bianchi groups. It is easy to see that Bianchi groups II-VII satisfy the assumptions of the theorem. In this case $N = \mathbb{R}^2$ and $H = \mathbb{R}$, the dual of N is $\hat{N} = \mathbb{R}^2$ and is given by

$$\hat{N} = \{e^{i(\check{k},\check{x})} \colon\ \check{x}, \check{k} \in \mathbb{R}^2\}.$$

Let $\imath_N : \mathbb{R}^2 \to G$ be the natural inclusion. The action of G on $\hat{N}$ is given by

$$g\xi_{\check{k}}(\check{x}) = \xi_{\check{k}}(\imath_N^{-1}(g^{-1}\imath_N(\check{x})g)).$$

All Bianchi solvable groups are homeomorphic to $\mathbb{R}^3$, and we may choose a global chart on them. In particular we choose one adapted to the semidirect structure $\mathbb{R}^2 \times_F \mathbb{R}$ presented in the previous section. Then the multiplication law in G is given by

$$(x, y, z)(x', y', z') = ((x, y) + F(z)(x', y'), z + z').$$

The unit $e \in G$ is given by $e = (0, 0, 0)$, and the inverse map by

$$(x, y, z)^{-1} = (-F^{-1}(z)(x, y), -z).$$

In particular, if $(\check{x}, 0) = (x, y, 0) \in \imath_N(\mathbb{R}^2)$ and $(g_x, g_y, g_z) \in G$, then

$$(g_x, g_y, g_z)^{-1}(x, y, 0)(g_x, g_y, g_z) = (F^{-1}(g_z)(x, y), 0),$$

that is, the conjugation map $n \to g^{-1}ng$ is given by $(x, y) \to F^{-1}(g_z)(x, y)$. Thus the action of G on $\hat{N}$ is

$$g\xi_{\check{k}}(\check{x}) = \xi_{\check{k}}(F^{-1}(g_z)\check{x}) = e^{i(\check{k},F^{-1}(g_z)\check{x})} = e^{i(F^{\perp}(g_z)\check{k},\check{x})},$$

where $F^{\perp}(g_z)$ is the inverse transpose of the matrix $F(g_z)$. This means that this action can be described by

$$g\breve{k} = F^{\perp}(g_z)\breve{k},\ g \in G,\ \breve{k} \in \mathbb{R}^2.$$

Denote by $V^0 \subset \mathbb{R}^2$ the eigenspace of $M^{\top}$ corresponding to the eigenvalue 0 (the null space). Then it will be also the joint eigenspace of the matrices $F^{\perp}(g_z) = e^{-g_z M^{\top}}$ corresponding to the eigenvalue 1 simultaneously for all $g_z \in \mathbb{R}$. Let us write the stabilizer condition,

$$e^{-g_z M^{\top}}\breve{k} = \breve{k}.$$

Then the stabilizer $G_{\breve{k}}$ and the little group $H_{\breve{k}}$ will be

$$G_{\breve{k}} = \imath_N(\mathbb{R}^2) \cdot H_{\breve{k}}$$

and

$$H_{\breve{k}} = \begin{cases} \mathbb{R} \text{ if } \breve{k} \in V^0, \\ \{0\} \text{ else.} \end{cases}$$

Define the following space of irreducible representations of G:

$$\hat{J} = (V^0 \times \mathbb{R}) \cup (\mathbb{R}^2 \setminus V^0).$$

For each $\mu \in \hat{J}$ the corresponding irreducible representation is given by

$$T_\mu(g) = e^{i(\breve{k},\breve{g})}e^{ik_3 g_3},\ \mu = (\breve{k}, k_3)$$

if $\mu \in V^0 \times \mathbb{R}$, and

$$T_\mu = T_{\breve{k}} = \mathbf{Ind}_{\mathbb{R}^2}^{G}(e^{i(\breve{k},\cdot)}),\ \mu = \breve{k},$$

if $\mu \in \mathbb{R}^2 \setminus V^0$. The orbit $\mathcal{O}_{\breve{k}}$ is $\{\breve{k}\}$ if $\breve{k} \in V^0$ and $F^{\perp}(\mathbb{R})\breve{k}$ otherwise. As mentioned in the theorem, two representations $\mu, \mu' \in \hat{J}$ are equivalent if and only if $\breve{k}$ and $\breve{k}'$ are on the same orbit, $\breve{k} = F^{\perp}(z)\breve{k}'$, and the corresponding representations of $H_{\breve{k}}$ and $H_{\breve{k}'}$ are equivalent when intertwined with the action of z. The first condition can be satisfied non-trivially if $\breve{k}, \breve{k}' \in \mathbb{R}^2 \setminus V^0$, but then $H_{\breve{k}} = H_{\breve{k}'} = \{0\}$, and thus there exists only the trivial representation $\rho = 1$. Thus representations $\mu, \mu' \in \mathbb{R}^2 \setminus V^0$ are equivalent if and only if they are on the same orbit. On the other hand, let $\mu, \mu' \in V^0$ such that $\breve{k} = \breve{k}'$, and the first condition is satisfied trivially. Then $G_{\breve{k}} = G$, and $G/G_{\breve{k}} = \{1\}$, so the action of 1 cannot intertwine inequivalent representations of $H_{\breve{k}}$. Thus $\mu \sim \mu'$ means $\mu = \mu'$. Therefore the dual space $\hat{G}$ of G will be

$$\hat{G} = (V^0 \times \mathbb{R}) \cup (\mathbb{R}^2 \setminus V^0)/F^{\perp}(\mathbb{R}).$$

The null spaces V^0. Finally let us find the eigenspaces V^0 for different Bianchi groups. By a calculation of eigenvectors and eigenvalues of M we obtain

$$V_I^0 = \mathbb{R}^2,\, V_{II}^0 = \mathbb{R} \oplus \{0\},\, V_{III}^0 = \{0\} \oplus \mathbb{R},$$

$$V_{IV}^0 = \{0\},\, V_V^0 = \{0\},\, V_{VII} = \{0\},$$

and

$$V_{VI} = \begin{cases} \{0\} \oplus \mathbb{R}, \text{ if } q = 0, \\ \{0\} \text{ else.} \end{cases}$$

Note that as always with solvable groups, the irreducible representations are either 1-dimensional or infinite dimensional.

To obtain explicit descriptions of the dual groups $\hat{G}$ for each Bianchi class we have to calculate the orbits $\mathcal{O}_{\check{k}} = F^{\perp}(\mathbb{R})\check{k}$ explicitly, which is done in the next section. Note that the entire construction could have been performed through the machinery of exponential solvable Lie groups developed in [10] and [9], where the problem is solved exhaustively. In particular, it was shown that (as adapted to our terminology) there exists a cross section $\tilde{K}$, an algebraic submanifold of $\mathbb{R}^2$ which crosses each *generic orbit* (i.e., an orbit of maximal dimension) exactly once, and thus parameterizes the infinite dimensional representations. Having explicitly calculated $\tilde{K}$ we find $\hat{G} = (V^0 \times \mathbb{R}) \cup \tilde{K}$. But the methods of [10] are extremely general and involve simple but lengthy algebraic calculations, this is why we have preferred the original topological Mackey constructions.

3.4 Co-adjoint orbits of Bianchi II-VII groups

The term co-adjoint orbits would probably suit better to the solvable Lie theoretical method of orbits as established by Kirillov and accomplished by Currey. At this point we afford a small digression to demonstrate the equivalence of that approach with that we have adopted.

The Kirillov approach. The Lie algebra $\mathfrak{g} = \mathbb{R}^2 \times_f \mathbb{R}$ of G is modelled on the vector space $\mathbb{R}^3$, and as such its dual space $\mathfrak{g}'$ is again isomorphic to $\mathbb{R}^3$. We will fix this isomorphism by choosing the basis in $\mathfrak{g}'$ dual to our adapted basis of $\mathfrak{g}$. With this identification the co-adjoint action of G on $\mathfrak{g}' = \mathbb{R}^3$ is given by the matrix field $Ad_g^* = Ad_g^{\perp}$,

$$Ad_g^* = \begin{pmatrix} F^{\perp}(g_z) & 0 \\ & 0 \\ (g_x, g_y)F^{\perp}(g_z)M^{\top} & 1 \end{pmatrix}.$$

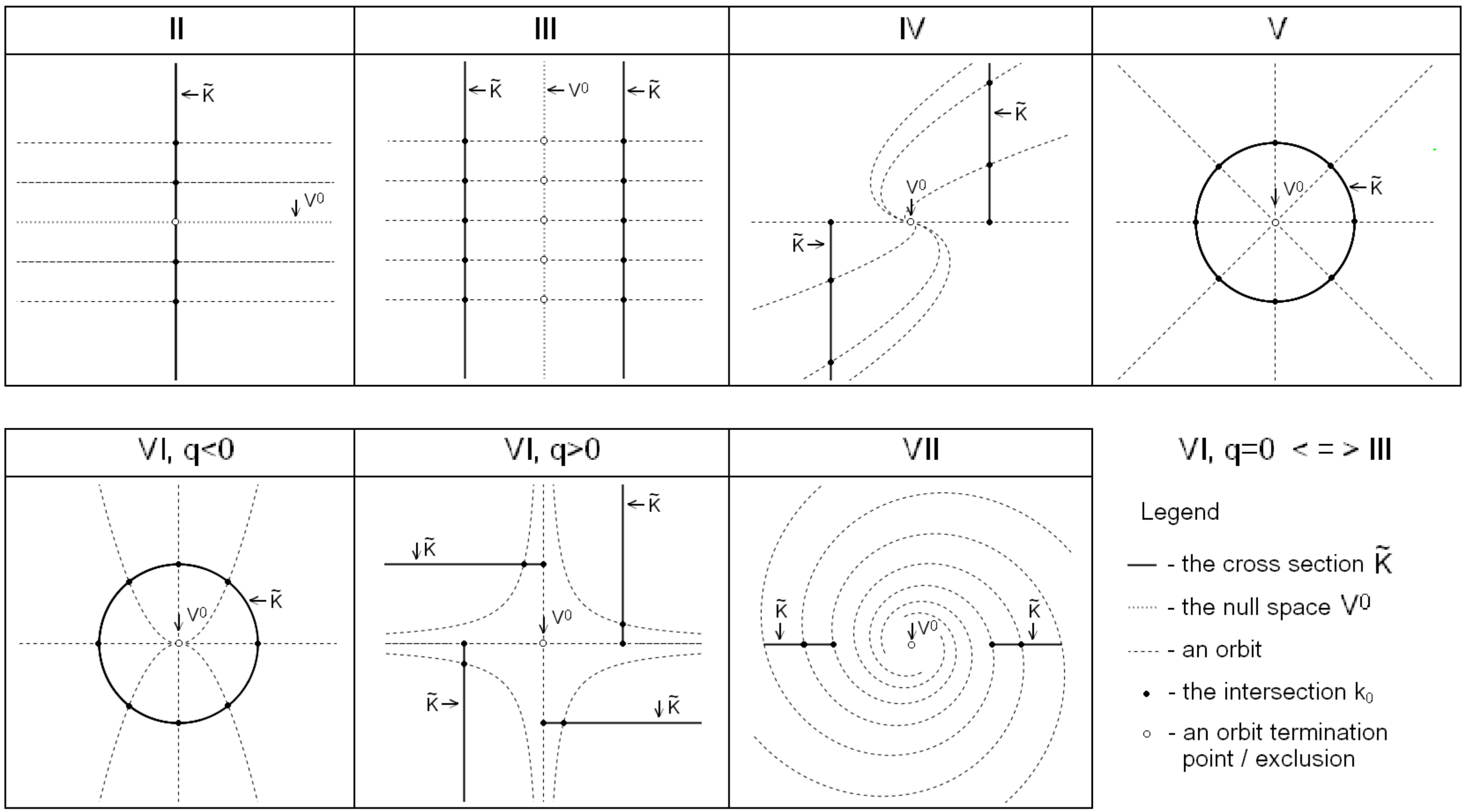
For any $\mathfrak{l} = (X^*, Y^*, Z^*) \in \mathfrak{g}'$ its orbit $\mathcal{O}_{\mathfrak{l}}$ is given by

$$\mathcal{O}_{\mathfrak{l}} = (F^{\perp}(\mathbb{R})(X^*, Y^*), (\mathbb{R}, \mathbb{R})F^{\perp}(\mathbb{R})M^{\top}(X^*, Y^*) + Z^*),$$

and the space of orbits $\{\mathcal{O}_{\mathfrak{l}}\}$ with the quotient topology induced from $\mathfrak{g}'$ is homeomorphic to $\hat{G}$ with the Fell topology [21]. One can easily see that the orbits are of two types: those of (X^*, Y^*, Z^*) with $(X^*, Y^*) \in V^0$ or $(X^*, Y^*) \notin V^0$. The formers are the so called *degenerate* orbits with dimension 0 (singletons), and the latters are the *generic* orbits with maximal dimension 3. This is exactly the same result we obtained above by Mackey machine.

The generic orbits and the cross sections. Here we will try to find the generic orbits $F^{\perp}(\mathbb{R})\breve{k}_0 \in \hat{G}$ mentioned in the previous section and corresponding cross-sections $\tilde{K} \in \mathbb{R}^2$. The latters will be algebraic manifolds composed of one or more connected components. In all cases V^0 is a subset of Lebesgue measure 0 in $\mathbb{R}^2$. By the definition of the cross section $\tilde{K}$, the submanifold $\mathbb{R}^2 \setminus V^0$ can be parameterized by a global chart $\breve{k} = \breve{k}(k, r)$, $(k, r) \in \mathfrak{K} \times \mathbb{R}$, such that $\breve{k}(k, r) = F^{\perp}(r)\breve{k}_0(k)$ and $\breve{k}_0(k) = \breve{k}(k, 0) \in \tilde{K}$. Under this diffeomorphism the Lebesgue measure $d\breve{k}$ becomes $\rho(k, r)dkdr$, where $\rho(k, r) = |\det \partial(\breve{k})/\partial(k, r)|$.

The co-adjoint orbits of Bianchi II-VII groups



Now let us proceed to the determination of the orbits and the cross sections case by case. The graph above illustrates them qualitatively.

II. We have

$$F^{\perp}(r)(k_x, k_y) = (k_x - rk_y, k_y),$$

hence the orbit through $\breve{k} \in \mathbb{R}^2 \setminus V^0$ is $F^{\perp}(\mathbb{R})(k_x, k_y) = (\mathbb{R}, k_y)$. The cross section can be chosen to be $\tilde{K} = \breve{k}_0(\mathfrak{K})$, $\mathfrak{K} = \mathbb{R} \setminus \{0\}$, $\breve{k}_0(k) = (0, k)$. Indeed, any orbit $(\mathbb{R}, k_y)$ meets $\tilde{K}$ exactly once at $\breve{k}_0(k_y)$. Then

$$\rho(k, r) = \left| \det \left(F^{\perp}(r) \frac{\partial \breve{k}_0(k)}{\partial k}, \frac{\partial F^{\perp}(r)}{\partial r} \breve{k}_0(k) \right) \right| = |k|.$$

III. In this case

$$F^{\perp}(r)(k_x, k_y) = (e^{-r} k_x, k_y),$$

and the orbit through $\breve{k} \in \mathbb{R}^2 \setminus V^0$ is $F^{\perp}(\mathbb{R})(k_x, k_y) = (sgn(k_x)\mathbb{R}_+, k_y)$. Let $\mathfrak{K} = \mathbb{R} \times \{-1, 1\}$, $k = (k_1, k_2)$. The cross section is the image $(-1, \mathbb{R}) \cup (1, \mathbb{R})$ of the map $\breve{k}_0(k) = (k_2, k_1)$. We find

$$\rho(k, r) = e^{-r}.$$

IV. For this group

$$F^{\perp}(r)(k_x, k_y) = (e^{-r} k_x - re^{-1} k_y, e^{-r} k_y),$$

and the orbits are complicated. We set $\mathfrak{K} = \mathbb{R}_{+0} \times \{-1, 1\}$, $k = (k_1, k_2)$ and $\breve{k}_0(k) = (k_2, sgn(k_2)k_1)$. That this is a cross section can be checked immediately. The measure density ρ is

$$\rho(k, r) = e^{-2r}(1 + k_1).$$

V. Now

$$F^{\perp}(r)(k_x, k_y) = e^{-r}(k_x, k_y),$$

and the orbits are simply the incoming radial rays. Set $\mathfrak{K} = \mathbb{R}/2\pi\mathbb{Z}$ and $\breve{k}_0(k) = (\cos(k), \sin(k))$. It follows

$$\rho(k, r) = e^{-2r}.$$

VI. For this group we consider only the case $q \neq 0$ as $q = 0$ is simply the group III.

$$F^{\perp}(r)(k_x, k_y) = (e^{-r}k_x, e^{qr}k_y),$$

and the orbits are incoming polynomial curves if $q < 0$ and hyperbolic curves if $q > 0$. For $q < 0$ set $\mathfrak{K} = \mathbb{R}/2\pi\mathbb{Z}$ and $\breve{k}_0(k) = (\cos(k), \sin(k))$. Then

$$\rho(k, r) = e^{-(1-q)r}(\cos^2(k) - q\sin^2(k)).$$

For $q > 0$ set $\mathfrak{K} = \mathbb{R}_{+0} \times \{0, 1, 2, 3\}$, $k = (k_1, k_2)$ and

$$\breve{k}_0(k) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{k_2} \begin{pmatrix} 1 \\ k_1 \end{pmatrix}.$$

Thus

$$\rho(k, r) = q^{k_2 \bmod 2} e^{-(1-q)r}.$$

VII. The co-adjoint action in this group is given by

$$F^{\perp}(r)(k_x, k_y) = e^{-pr}(k_x \cos r - k_y \sin r, k_x \sin r + k_y \cos r),$$

and the orbits are incoming or outgoing spirals depending on whether $p < 0$ or $p > 0$. We take $\mathfrak{K} = (-e^{\pi p}; -1] \cup [1, e^{\pi p})$ and $\breve{k}_0(k) = (k, 0)$. Each orbit clearly intersects $\tilde{K}$ exactly once. Finally

$$\rho(k, r) = e^{-2pr}|k|.$$

Note that in all cases we have chosen $\tilde{K}$ such that it possesses an involution $\breve{k}_0(-k) = -\breve{k}_0(k)$, which will be useful in later constructions. Of course, these choices of cross sections are not unique, neither they need to correspond to those suggested by Currey theory. In fact, one may make any other choice for convenience and calculate the corresponding measure density ρ precisely as we did.

3.5 The explicit Plancherel formula for Bianchi II-VII groups

We will obtain the Plancherel measure by extending the idea suggested in [21] for Heisenberg groups to all solvable Bianchi groups. Namely, we will exploit the Euclidean Parseval equality on the homeomorphic space $\mathbb{R}^3$.

Introductory material. Before going to the solvable groups II-VII let us recall the well-known form of the Plancherel formula for the Abelian group $\mathbb{R}^3$. The Fourier transform of a function $f \in C_0^\infty(\mathbb{R}^3)$ is defined by

$$\hat{f}(\vec{k}) = \int_{\mathbb{R}^3} d\vec{x} e^{-i(\vec{k},\vec{x})} f(\vec{x}),$$

and the Plancherel formula is

$$\int_{\mathbb{R}^3} d\vec{x} |f(\vec{x})|^2 = (2\pi)^3 \int_{\mathbb{R}^3} d\vec{k} |\hat{f}(\vec{k})|^2.$$

The Plancherel measure is simply $d\nu(\vec{k}) = (2\pi)^3 d\vec{k}$, proportional to the Lebesgue measure on $\mathbb{R}^3$.

We start by noting that being an algebraic (matrix) group G is necessarily type I (Theorem 7.8 or 7.10 [21]), and the normal subgroup N is unimodular and therefore in the kernel of the modular function Δ. It follows from (Theorem 7.6 [21]) that the Mackey Borel structure on $\hat{G}$ is standard, and thereby due to (Lemma 7.39 [21]) we have a measurable field of representations π_p on $p \in \hat{G}$, such that $\pi_p \in p$ (or equivalently, we have a measurable choice of representatives of each equivalence class $[\pi] \in \hat{G}$.) Henceforth we will speak of a representation $\pi \in \hat{G}$ meaning the value of this field at a given point $[\pi] \in \hat{G}$. As can be inferred from [9] in the language of solvable Lie groups, only those irreducible representations corresponding to the generic orbits (i.e., orbits of maximal dimension) deserve non-zero Plancherel measure. Therefore only T_μ with $\mu \in \mathbb{R}^2 \setminus V^0$ (generic representations) will play a role in the Fourier transform. We proceed to their construction as $T_{\breve{k}} = \mathbf{Ind}_{\mathbb{R}^2}^G(e^{i(\breve{k},.)})$ following (§6.1 [21]).

The Fourier transform at generic representations. For each $\breve{k} \in \mathbb{R}^2 \setminus V^0$ the representation Hilbert space $H_{\breve{k}}$ of $\nu_{\breve{k}} = e^{i(\breve{k},.)}$ is $H_{\breve{k}} = \mathbb{C}$. The homogeneous space

$G/N = \mathbb{R}$ has a natural G-invariant measure, which is the Lebesgue measure dz. The representation space of T_μ is then the completion $L^2(\mathbb{R}, \mathbb{C})$ of the space of compactly supported continuous sections in homogeneous Hermitian line bundle $\mathbb{R} \times \mathbb{C}$, and the action of G on it is given by

$$T_{\check{k}}(g)f[z] = e^{-i(\check{k},(g^{-1}z)_N)}f[(g^{-1}z)_H] = e^{i(\check{k},F(-z)\check{g})}f[z - g_z],\ g = (\check{g}, g_z) \in G,\ f \in C_0(\mathbb{R}, \mathbb{C}),$$

where for any $g \in G$ we write $g = g_N g_H$, $g_N \in N$, $g_H \in H$. For $f \in C_0(G)$ define the (harmonic analytical) Fourier transform by

$$\hat{f}(\pi) = \pi(f)D_\pi = \int_G f(g)\pi(g)D_\pi dg,$$

where the operator D_π is defined on $\phi \in C_0(\mathbb{R}, \mathbb{C})$ by

$$D_\pi \phi[z] = \Delta(z)^{+\frac{1}{2}}\phi[z] = (\det F(z))^{-\frac{1}{2}}\phi[z].$$

(Note that there is a misprint in the formula (7.49) of [21], and the sign $-$ in the power of Δ should be replaced by $+$. The author confirmed this in a private communication.) By (Theorem 7.50 [21]) the operator fields $\hat{f}(\pi)$ are measurable fields of Hilbert-Schmidt operators, and if we identify the space of Hilbert-Schmidt operators on H_π with the tensor product space $H_\pi \otimes H_\pi^*$ then the Fourier transform gives an isomorphism

$$L^2(G) \sim \int_{\hat{G}}^{\oplus} d\nu(\pi) H_\pi \otimes H_\pi^*.$$

To find the Plancherel measure $d\nu(\pi)$ we calculate the Fourier transforms $\hat{f}(T_{\check{k}})$ directly. For $\phi \in C_0(\mathbb{R}, \mathbb{C})$ we have

$$\hat{f}(T_{\check{k}})\phi[r] = \int_G f(g)T_{\check{k}}(g)D_\pi\phi[r]dg =$$

$$= \mathfrak{h}\int_{\mathbb{R}^3} dg_x dg_y dg_z f(g_x, g_y, g_z)e^{i(\check{k},F(-r)\check{g})}(\det F(g_z))^{-1}(\det F(r - g_z))^{-\frac{1}{2}}\phi[r - g_z] =$$

by a substitution $g'_z = r - g_z$

$$= \mathfrak{h}\int_{\mathbb{R}^3} dg_x dg_y dg'_z f(g_x, g_y, r - g'_z)e^{i(\check{k},F(-r)\check{g})}(\det F(r - g'_z))^{-1}(\det F(g'_z))^{-\frac{1}{2}}\phi[g'_z] =$$

$$= \mathfrak{h}\int_{\mathbb{R}^3} dg_x dg_y dg'_z f(g_x, g_y, r - g'_z)e^{i(F^{\perp}(r)\check{k},\check{g})}(\det F(r - g'_z))^{-1}(\det F(g'_z))^{-\frac{1}{2}}\phi[g'_z].$$

Thus $\hat{f}(T_{\breve{k}})$ is an integral operator with a smooth kernel

$$\mathcal{K}^f_{\breve{k}}(r, g'_z) = \mathfrak{h}\bar{\mathfrak{F}}_{\mathbb{R}^2}[f(.,.,r-g'_z)](F^{\perp}(r)\breve{k})(\det F(r-g'_z))^{-1}(\det F(g'_z))^{-\frac{1}{2}},$$

where

$$\bar{\mathfrak{F}}_{\mathbb{R}^2}[\psi(.,.)](\breve{k}) = \int_{\mathbb{R}^2} dxdy\psi(x,y)e^{i(\breve{k},\breve{x})}.$$

The Hilbert-Schmidt norm $|||\hat{f}(T_{\breve{k}})|||$ is given by

$$|||\hat{f}(T_{\breve{k}})|||^2 = \mathfrak{h}^2 \int_{\mathbb{R}^2} drdg'_z|\mathcal{K}^f_{\breve{k}}(r, g'_z)|^2.$$

Coming back to the original variable $g_z = r - g'_z$

$$|||\hat{f}(T_{\breve{k}})|||^2 = \mathfrak{h}^2 \int_{\mathbb{R}^2} drdg_z \left|\bar{\mathfrak{F}}_{\mathbb{R}^2}[f(.,.,g_z)](F^{\perp}(r)\breve{k})\right|^2 (\det F(g_z))^{-2}(\det F(r-g_z))^{-1} =$$

$$= \mathfrak{h}^2 \int_{\mathbb{R}^2} drdg_z \left|\bar{\mathfrak{F}}_{\mathbb{R}^2}[f(.,.,g_z)](F^{\perp}(r)\breve{k})\right|^2 (\det F(g_z))^{-1}(\det F(r))^{-1} =$$

by Fubini's theorem,

$$= \mathfrak{h}^2 \int_{\mathbb{R}} dg_z(\det F(g_z))^{-1} \int_{\mathbb{R}} dr \left|\bar{\mathfrak{F}}_{\mathbb{R}^2}[f(.,.,g_z)](F^{\perp}(r)\breve{k})\right|^2 (\det F(r))^{-1}.$$

The Plancherel formula. Now we refer to the previous section about the co-adjoint orbits, and note, that in all cases $\rho(k,r) = \dot{\nu}(k)(\det F(r))^{-1}$ with some continuous non-negative function $\dot{\nu}(k)$ on $\tilde{K}$. We will shortly see that

$$d\nu(k) = \mathfrak{h}^{-1}\dot{\nu}(k)dk \tag{3.1}$$

is exactly the Plancherel measure desired. Indeed,

$$\int_{\tilde{K}} dk\mathfrak{h}^{-1}\dot{\nu}(k)|||\hat{f}(T_{\breve{k}_0(k)})|||^2 = \mathfrak{h}\int_{\tilde{K}} dk\dot{\nu}(k)\int_{\mathbb{R}} dg_z(\det F(g_z))^{-1}...$$

$$...\int_{\mathbb{R}} dr \left|\bar{\mathfrak{F}}_{\mathbb{R}^2}[f(.,.,g_z)](F^{\perp}(r)\breve{k}_0(k))\right|^2 (\det F(r))^{-1} =$$

by another application of Fubini's theorem ([16],chapter XIII),

$$= \mathfrak{h}\int_{\mathbb{R}} dg_z(\det F(g_z))^{-1}\int_{\tilde{K}} dk\int_{\mathbb{R}} dr\rho(k,r)\left|\bar{\mathfrak{F}}_{\mathbb{R}^2}[f(.,.,g_z)](F^{\perp}(r)\breve{k}_0(k))\right|^2 =$$

by definition of $\rho(k,r)$,

$$= \mathfrak{h}\int_{\mathbb{R}} dg_z(\det F(g_z))^{-1}\int_{\mathbb{R}^2} d\check{k}\left|\bar{\mathfrak{F}}_{\mathbb{R}^2}[f(.,.,g_z)](\check{k})\right|^2 =$$

by Euclidian Parseval formula,

$$= \mathfrak{h}\int_{\mathbb{R}} dg_z(\det F(g_z))^{-1}\int_{\mathbb{R}^2} dg_x dg_y\,|f(g_x,g_y,g_z)|^2 = \int_G dg|f(g)|^2,$$

thus we arrive at an explicit Plancherel forumla,

$$\int_{\check{K}} d\nu(k)|||\hat{f}(T_{\check{k}_0(k)})|||^2 = \int_G dg|f(g)|^2.$$

The Plancherel measures for groups II-VII are thus given by

$$\dot{\nu}_{II}(k) = |k|,\ \dot{\nu}_{III}(k) = 1,\ \dot{\nu}_{IV}(k) = 1 + k_1,\ \dot{\nu}_V(k) = 1,\ \dot{\nu}_{VI-}(k) = \cos^2(k) - q\sin^2(k)$$

$$\dot{\nu}_{VI+}(q) = q^{k_2 \bmod 2},\ \dot{\nu}_{VII}(k) = |k|.$$

Note that we could have chosen the cross section for VI, $q<0$ the same as that for VI, $q>0$ and get a uniform Plancherel measure $\dot{\nu}_{VI} = \dot{\nu}_{VI_+}$ for all Bianchi VI groups, but we preferred the more conventional circle to the strange quartet of rays when it was possible. This can be altered for any technical purposes when needed.

3.6 Scalar spectral analysis on Bianchi I-VII groups

Here the term scalar spectral analysis is understood as the spectral theory of the scalar Laplacian. Of course, there is no distinguished Laplacian on these groups. We will consider *any Laplacian* which arises as the metric operator with respect to any conserved metric on the group.

Let G be one of these groups, and let $\mathfrak{L}(G)$ be its Lie algebra generated by three right invariant vector fields ξ_1,ξ_2,ξ_3. Let further X_1,X_2,X_3 be a basis of left invariant vector fields on G, and $d\omega^1, d\omega^2, d\omega^3$ the dual basis. Any left invariant metric h_{ab} on G can be written as

$$h_{ab} = \sum_{i,j=1}^{3} \check{h}_{ij} d\omega^i_a d\omega^j_b,$$

where $\check{h}_{ij}$ is any symmetric positive definite 3×3 matrix, and the corresponding metric Laplacian will be

$$\boldsymbol{\Delta}_h = \sum_{i,j=1}^{3} \check{h}^{ij} X_i X_j, \tag{3.2}$$

with $\check{h}^{ij} = (\check{h}_{ij})^{-1}$. To see this first note that

$$\sum_{i,j=1}^{3} \check{h}^{ij} X_i X_j f = \sum_{i,j=1}^{3} \check{h}^{ij} \sum_{l,m=1}^{3} [X_i^l X_j^m \partial_l \partial_m + X_i^l \partial_l X_j^m \partial_m] f.$$

On the other hand the connection Laplacian related to the Levi-Civita connection is given by

$$\boldsymbol{\Delta}_h = \sum_{i,j=1}^{3} \check{h}^{ij} \sum_{l,m=1}^{3} [X_i^l X_j^m \partial_l \partial_m - \sum_{k=1}^{3} X_i^l X_j^m \boldsymbol{\Gamma}_{lm}^k \partial_k],$$

where $\boldsymbol{\Gamma}_{lm}^k$ is the Christoffel symbol. This together with the observation

$$\sum_{l,m=1}^{3} X_i^l X_j^m \boldsymbol{\Gamma}_{lm}^k = -\frac{1}{2}(X_i^l \partial_l X_j^m + X_j^l \partial_l X_i^m),$$

which follows from $\nabla_{X_i} X_j = \frac{1}{2}[X_i, X_j]$, gives (Eq.3.2). Our aim will be to find the eigenfunctions and the spectrum of $\boldsymbol{\Delta}_h$. If ξ_1 and ξ_2 commute and also commute with all X_i, then $\xi_1, \xi_2, \boldsymbol{\Delta}_h$ are a triple of commuting operators, and have common eigenfunctions. We will find those eigenfunctions and show that they are complete in the sense we desire. For the ease of notation let us denote $\check{h}^{2\times 2} = \check{h}^{ij}|_{i,j<3}$, $\check{h}^{\cdot 3} = \check{h}^{ij}|_{i<j=3}$ and $\check{h}^{3\cdot} = \check{h}^{ij}|_{j<i=3}$.

First let us describe the spectrum $\mathrm{Spec}(\boldsymbol{\Delta}_h)$ of the Laplacian $\boldsymbol{\Delta}_h$. We note, that $\boldsymbol{\Delta}_h$ is a negative semidefinite operator, as $(\boldsymbol{\Delta}_h f, f)_{L^2(G)} = -(d_h f, d_h f)_{L^2(G)} \leq 0$, where d_h is the exterior derivative with respect to the metric h_{ab}. Thus $\boldsymbol{\Delta}_h$ is a semibounded and real symmetric operator on $L^2(G)$. There are several ways of extending $\boldsymbol{\Delta}_h$ to a self-adjoint operator on $L^2(G)$. A real symmetric operator has a self-adjoint extension by von Neumann's theorem [49]. A semibounded symmetric operator has a self-adjoint by Friedrich's extension theorem [49]. But we have something stronger. The Lie group G with its left invariant Riemannian metric h_{ab} is a complete Riemannian manifold [40]. Then following [8] $\boldsymbol{\Delta}_h$ is essentially self-adjoint. Being a negative self-adjoint operator $\boldsymbol{\Delta}_h$ has a real non-positive spectrum, $\mathrm{Spec}(\boldsymbol{\Delta}_h) \subset (-\infty; 0]$. The semidirect structure of our groups satisfies the conditions of Lemma 5.6 of [40], and we have for the scalar curvature R_h the following formula,

$$R_h = -Tr[S^2] - (Tr[S])^2,$$

where we took into account that the normal Lie subgroup $\mathbb{R}^2$ with the induced metric is flat. The matrix S is given by

$$S = \frac{1}{2}(ad_{(0,0,1)}|_{\mathbb{R}^2} + ad_{(0,0,1)}|^*_{\mathbb{R}^2}) = \frac{1}{2}(f(1) + f(1)^*),$$

where the adjoint * is understood as

$$h(Af, g) = h(f, A^* g),\ \forall f, g \in \mathfrak{L}(G),\ A \in Aut(\mathfrak{L}(G)).$$

Thus all our groups endowed with any left invariant Riemannian metric are spaces of constant negative curvature equal to R_h, which is given explicitly in terms of the matrices $f(1) = M$ and $\check{h}^{2\times 2}$ with $S = \frac{1}{2}(M + (\check{h}^{2\times 2})^{-1} M \check{h}^{2\times 2})$. This in turn implies following [19] that the essential spectrum of $\mathbf{\Delta}_h$ is precisely $\mathrm{EssSpec}(\mathbf{\Delta}_h) = (-\infty; R_h]$. For the group Bianchi I, all irreps are 1-dimensional, and as we will see later in the section, each eigenspace representation includes an infinite number of them, thus there is no discrete spectrum. For the remaining groups, in the previous section we have seen, that no finite dimensional representation enters the Plancherel formula. On the other hand, in the next section we will see, that the infinite dimensional eigenspaces exhaust $L^2(G)$, hence no finite dimensional eigenspace exists, i.e., the discrete spectrum is empty, $\mathrm{Spec}(\mathbf{\Delta}_h) = \mathrm{EssSpec}(\mathbf{\Delta}_h)$.

To find the generators ξ_i for Bianchi I-VII groups we differentiate the left translation map $\vec{x} \to g\vec{x}$,

$$g(x, y, z) = ((g_x, g_y) + F(g_z)(x, y), g_z + z),$$

and obtain

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ (x, & y)\dot{F}^\top(0) & 1 \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}.$$

We see that $\xi_1 = \partial_x$ and $\xi_2 = \partial_y$ do indeed commute. To find the left invariant vectors X_i (which are the generators of right translations) we differentiate the right translation map $\vec{x} \to \vec{x}g$,

$$(x, y, z)g = ((x, y) + F(z)(g_x, g_y), z + g_z),$$

and get

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} F^\top(z) & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}. \tag{3.3}$$

Thus ξ_1, ξ_2 do commute with all X_i. Now let $\zeta(\vec{x}) \in C^\infty(G)$ be a joint eigenfunction for $\{\xi_1, \xi_2, \mathbf{\Delta}_h\}$. Then it is necessarily of the form

$$\zeta(\vec{x}) = e^{i(\breve{k}_{\mathbb{C}}, \breve{x})} P(z),$$

where $\breve{k}_{\mathbb{C}} \in \mathbb{C}^2$, $\breve{x} = (x, y)$, and satisfies

$$\mathbf{\Delta}_h \zeta(\vec{x}) = \lambda \zeta(\vec{x}),$$

for some $\lambda \in \mathbb{C}$. A matrix representation of (Eq.3.2) and a bit of manipulation yields the following equation

$$\breve{h}^{33}\ddot{P}(z) + i(\breve{k}_{\mathbb{C}}^\top F(z)[\breve{h}^{\cdot 3} + (\breve{h}^{3\cdot})^\top])\dot{P}(z) - (\lambda + \breve{k}_{\mathbb{C}}^\top F(z)\breve{h}^{2\times 2}F^\top(z)\breve{k}_{\mathbb{C}} - i\breve{h}^{3\cdot}F^\top(z)M^\top\breve{k}_{\mathbb{C}})P(z) = 0,$$

where $\dot{F}^\top(z) = \partial_z e^{zM^\top} = F^\top(z)M^\top$ was used. This is a generalized time-dependent harmonic oscillator equation, which always have solutions, and those solutions comprise a two complex dimensional space. For given λ and $\breve{k}_{\mathbb{C}}$ let choose $P_{\lambda,k_{\mathbb{C}}}(z)$ and $Q_{\lambda,k_{\mathbb{C}}}(z)$ to be two linearly independent solutions (the choice of initial data may be arbitrary).

First we consider the group Bianchi I. Here $M = 0$, $F(z) = 1$ and the equation becomes

$$\breve{h}^{33}\ddot{P}(z) + i(\breve{k}_{\mathbb{C}}^\top[\breve{h}^{\cdot 3} + (\breve{h}^{3\cdot})^\top])\dot{P}(z) - (\lambda + \breve{k}_{\mathbb{C}}^\top\breve{h}^{2\times 2}\breve{k}_{\mathbb{C}})P(z) = 0.$$

One can easily check that $P(z) = e^{ik_z \cdot z}$ is a solution if

$$\lambda = -\vec{k}_{\mathbb{C}}^\top \breve{h}^{ij}\vec{k}_{\mathbb{C}},$$

where $\vec{k}_{\mathbb{C}} = (\breve{k}_{\mathbb{C}}, k_z)$. This is a consequence of the fact that for this group ξ_3 also commutes with all ξ_i and X_i, so that there exist joint eigenfunctions of the commuting operators $\xi_1, \xi_2, \xi_3, \mathbf{\Delta}_h$ of the form

$$\zeta(\vec{x}) = e^{i(\vec{k}_{\mathbb{C}}, \vec{x})},$$

corresponding to the eigenvalues

$$\lambda = -\vec{k}_{\mathbb{C}}^{\top} \check{h}^{ij} \vec{k}_{\mathbb{C}}.$$

In particular, when we restrict ourselves to the irreducibles $\vec{k}_{\mathbb{C}} = \vec{k} \in \mathbb{R}^3$, we obtain

$$\zeta_{\vec{k}}(\vec{x}) = e^{i(\vec{k},\vec{x})},$$

and immediately observe, that each eigenspace corresponding to the eigenvalue λ includes infinitely many $\vec{k}$-s, which satisfy

$$\lambda = -\vec{k}^{\top} \check{h}^{ij} \vec{k}.$$

Of course, these $e^{ik_z \cdot z}$ do not exhaust all solutions $P(z)$. But it turns out that $\zeta_{\vec{k}}$ constructed in this way are already complete in $L^2(G)$. Indeed, that is the essence of the Euclidean Parseval equality. To be more precise, we need to take $d\nu(\vec{k}) = \frac{1}{\mathfrak{h}} d\vec{k}$ as the Plancherel measure for the Euclidean Plancherel formula to hold. Equivalently we can renormalize $\zeta_{\vec{k}}$ by taking

$$\zeta_{\vec{k}}(\vec{x}) = \frac{1}{\sqrt{\mathfrak{h}}} e^{i(\vec{k},\vec{x})}$$

so that the Plancherel measure is independent of the metric. But this ease of construction is a peculiarity which the remaining groups Bianchi II-VII do not share, and we proceed to determine their eigenfunctions.

For the groups II-VII let us now restrict to $0 > \lambda \in \mathbb{R}$ and $\check{k}_{\mathbb{C}} = F^{\perp}(r) k_0(-k) \in \mathbb{R}^2 \setminus V^0$, $k \in \mathfrak{K}$, $r \in \mathbb{R}$ (minus sign for convenience). The equation now becomes

$$\begin{aligned} \check{h}^{33} \ddot{P}(z) + i(\check{k}_0(-k)^{\top} F(z-r)[\check{h}^{\cdot 3} + (\check{h}^{3\cdot})^{\top}]) \dot{P}(z) - \\ (\lambda + \check{k}_0(-k)^{\top} F(z-r) \check{h}^{2\times 2} F^{\top}(z-r) \check{k}_0(-k) - i\check{h}^{3\cdot} F^{\top}(z-r) M^{\top} \check{k}_0(-k)) P(z) = 0, \end{aligned} \tag{3.4}$$

and the two independent solutions will be denoted by $P_{\lambda,k,r}(z)$ and $Q_{\lambda,k,r}(z)$. If we denote $P_{\lambda,k,0}(z) = P_{\lambda,k}(z)$, $Q_{\lambda,k,0}(z) = Q_{\lambda,k}(z)$, then a variable substitution $z - r \to z$ shows that we can choose $P_{\lambda,k,r}(z) = P_{\lambda,k}(z-r)$, $Q_{\lambda,k,r}(z) = Q_{\lambda,k}(z-r)$. Another thing that can be noticed in equation (Eq.3.4) by taking the complex conjugate is that we can choose $P_{\lambda,-k}(z) = \bar{P}_{\lambda,k}(z)$, $Q_{\lambda,-k}(z) = \bar{Q}_{\lambda,k}(z)$. Finally we construct the eigenfunctions

$$\zeta_{k,\lambda,r,s}(\vec{x}) = (\det F(-r)) e^{i(F^{\perp}(r)\check{k}_0(-k),\check{x})} P_{\lambda,k,s}(z-r), \tag{3.5}$$

where to $s = 1$ (-1) corresponds $P_{\lambda,k,s} = P_{\lambda,k}$ $(Q_{\lambda,k})$. Note that each $\zeta_{k,\lambda,r,s}$ enters with its conjugate, $\bar{\zeta}_{k,\lambda,r,s} = \zeta_{-k,\lambda,r,s}$. As we will see in the next section, $P_{\lambda,k,s}$ are orthogonal with respect to the weight $\det F(-z)$, which shows that $\zeta_{k,\lambda,r,s}$ just defined are orthogonal with respect to the same weight. Again, instead of using the Plancherel measure (Eq.3.1) we can use $d\nu(k) = \dot{\nu}(k)dk$ but renormalize

$$\zeta_{k,\lambda,r,s}(\vec{x}) = \frac{1}{\sqrt{\mathfrak{h}}}(\det F(-r))e^{i(F^{\perp}(r)\breve{k}_0(-k),\breve{x})}P_{\lambda,k,s}(z-r).$$

Note that by (Eq.3.3) the number $\mathfrak{h}$ is nothing else but $\sqrt{\det \breve{h}_{ij}}$.

3.7 Fourier transform on Bianchi II-VII groups

As a first step on the way of showing the completeness of $\{\zeta_{k,\lambda,r,s}\}$ we prove a small and easy proposition. Denote the differential operator

$$D_{\breve{k}} = \breve{h}^{33}\frac{d^2}{dz^2}+i(\breve{k}^{\top}F(z)[\breve{h}^{\cdot 3}+(\breve{h}^{3\cdot})^{\top}])\frac{d}{dz}-(\breve{k}^{\top}F(z)\breve{h}^{2\times 2}F^{\top}(z)\breve{k}-i\breve{h}^{3\cdot}F^{\top}(z)M^{\top}\breve{k}),\ \breve{k}\in\mathbb{R}^2\backslash V^0,$$

which by definition satisfies

$$D_{\breve{k}}f(z) = e^{-i(\breve{k},\breve{x})}\mathbf{\Delta}_h\left[e^{i(\breve{k},\breve{x})}f(z)\right],\ f\in C_0^{\infty}(\mathbb{R}).$$

Proposition 3.1 *The operator $D_{\breve{k}}$ is formally self-adjoint on $L^2(\mathbb{R}, \det F(-z)dz)$ for any $\breve{k}\in\mathbb{R}^2\setminus V^0$.*

Proof: Let us first write the Green's identity for the operator $\mathbf{\Delta}_h$ on the infinite tube $D^1\times\mathbb{R}\subset G$ where D^1 is the unit disk in the $\breve{x}$-plane,

$$\int_{D^1\times\mathbb{R}} d\vec{x}\left(e^{-i(\breve{k},\breve{x})}\bar{g}(z)\mathbf{\Delta}_h\left[e^{i(\breve{k},\breve{x})}f(z)\right]-\mathbf{\Delta}_h\left[e^{-i(\breve{k},\breve{x})}\bar{g}(z)\right]e^{i(\breve{k},\breve{x})}f(z)\right)=$$

$$=\int_{S^1\times\mathbb{R}} dzdl(\breve{x})\left(e^{-i(\breve{k},\breve{x})}\bar{g}(z)(\breve{x},\frac{\partial}{\partial\breve{x}})\left[e^{i(\breve{k},\breve{x})}f(z)\right]-(\breve{x},\frac{\partial}{\partial\breve{x}})\left[e^{-i(\breve{k},\breve{x})}\bar{g}(z)\right]e^{i(\breve{k},\breve{x})}f(z)\right)=$$

$$=\int_{S^1\times\mathbb{R}} dzdl(\breve{x})2i\bar{g}(z)f(z)(\breve{x},\breve{k})=0.$$

Next we note that

$$\int_{D^1\times\mathbb{R}} d\vec{x}\left(e^{-i(\check{k},\check{x})}\bar{g}(z)\mathbf{\Delta}_h\left[e^{i(\check{k},\check{x})}f(z)\right]-\mathbf{\Delta}_h\left[e^{-i(\check{k},\check{x})}\bar{g}(z)\right]e^{i(\check{k},\check{x})}f(z)\right)=$$

$$=\int_{D^1\times\mathbb{R}} dxdydz(\det F(-z))\left(\bar{g}(z)D_{\check{k}}f(z)-\bar{D}_{\check{k}}[\bar{g}(z)]f(z)\right)=$$

$$=\pi\int_{\mathbb{R}} dz(\det F(-z))\left(\bar{g}(z)D_{\check{k}}f(z)-\bar{D}_{\check{k}}[\bar{g}(z)]f(z)\right)=0,$$

which holds on the dense subset $C_0^\infty(\mathbb{R})$ of $L^2(\mathbb{R},\det F(-z)dz)$, and the formal self-adjointness is thus proven.$\Box$

Now from the definition it is clear, that $D_{\check{k}}$ is a negative definite operator (because $\mathbf{\Delta}_h$ is such), and is hence upper semibounded, and has a self-adjoint extension to entire $L^2(\mathbb{R},\det F(-z)dz)$ by Friedrichs extension theorem [49]. In particular, for $\check{k}=\check{k}_0(-k)$, $k\in\mathfrak{K}$, the generalized eigenfunctions $\{P_{\lambda,k,s}\}_{\lambda\in Sp(\mathbf{\Delta}_h),s=\pm1}$ are complete and give rise to a Fourier transform $\mathfrak{F}_{\check{k}_0(-k)}$ on $L^2(\mathbb{R},\det F(-z)dz)$ by means of an abstract eigenfunction expansion (note that each eigenspace is 2-dimensional and thus our earlier theory of chapter 1 applies). $\mathfrak{F}_{\check{k}_0(-k)}$ is given by

$$(\mathfrak{F}_{\check{k}_0(-k)}f)(\lambda,s)=\int_{\mathbb{R}} dz(\det F(-z))\bar{P}_{\lambda,k,s}(z)f(z).$$

Define now the linear isomorphism $\mathfrak{V}:L^2(\mathbb{R})\to L^2(\mathbb{R},\det F(-z)dz)$ by setting $f=\mathfrak{V}\phi$ if $f(z)=\phi(-z)(\det F(z))^{\frac{1}{2}}$. This induces a Fourier transform $\mathfrak{F}_k=\mathfrak{F}_{\check{k}_0(-k)}\mathfrak{V}$ which acts as

$$(\mathfrak{F}_k\phi)(\lambda,s)=\int_{\mathbb{R}} dz(\det F(z))^{\frac{1}{2}}P_{\lambda,k,s}(-z)\phi(z)\doteq\tilde{\phi}(\lambda,k,s).$$

The inversion formula will be

$$\phi(z)=(\det F(z))^{\frac{1}{2}}\sum_{s=\pm1}\int_{Sp(\mathbf{\Delta}_h)} d\lambda\tilde{\phi}(\lambda,k,s)\bar{P}_{\lambda,k,s}(-z).$$

Now we are in the position to show how $\zeta_{k,\lambda,r,s}$ are related to the irreducible representations $T_{\check{k}_0(k)}$. Consider the following transformation on $f\in C_0^\infty(G)$,

$$\tilde{f}(k,\lambda,r,s)=\int_G dg\bar{\zeta}_{k,\lambda,r,s}(g)f(g).$$

We will see that $\tilde{f}(k,\lambda,r,s)$ are in some sense proportional to the matrix columns of the operators $\hat{f}(T_{\check{k}_0(k)})$. First we see that

$$\tilde{f}(k,\lambda,r,s)=\mathfrak{h}(\det F(-r))\int_{\mathbb{R}^3}dxdydz(\det F(-z))f(x,y,z)e^{i(F^{\perp}(r)\check{k}_0(k),\check{x})}\bar{P}_{\lambda,k,s}(-(r-z))=$$

$$=\mathfrak{h}(\det F(-r))\int_{\mathbb{R}^3}dxdydz(\det F(-z))(\det F(r-z))^{-\frac{1}{2}}f(x,y,z)e^{i(F^{\perp}(r)\check{k}_0(k),\check{x})}\times$$

$$\times(\det F(r-z))^{\frac{1}{2}}\bar{P}_{\lambda,k,s}(-(r-z)).$$

Next we recognize that this is related to the extension of the operator $\hat{f}(T_{\check{k}_0(k)})$ from $L^2(\mathbb{R})$ to $C^\infty(\mathbb{R})$,

$$\tilde{f}(k,\lambda,r,s)=(\det F(-r))\hat{f}(T_{\check{k}_0(k)})\left[(\det F(r))^{\frac{1}{2}}\bar{P}_{\lambda,k,s}(-r)\right].$$

Integrating we obtain

$$\sum_{s=\pm1}\int_{Sp(\mathbf{\Delta}_h)}d\lambda\tilde{f}(k,\lambda,r,s)\tilde{\phi}(\lambda,k,s)=(\det F(-r))\hat{f}(T_{\check{k}_0(k)})\phi[r].\tag{3.6}$$

Recall now the Fourier inversion formula as given in [9] (his notations are a bit different, and we have adapted them to ours, which we adopted from [21]),

$$f(1)=\int_{\mathfrak{K}}d\nu(k)Tr\left[D_\pi\hat{f}(T_{\check{k}_0(k)})\right].\tag{3.7}$$

Formally a matrix element of $D_\pi\hat{f}(T_{\check{k}_0(k)})$ would be an expression

$$\left((\det F(z))^{\frac{1}{2}}\bar{P}_{\lambda',k,s'}(-z),D_\pi\hat{f}(T_{\check{k}_0(k)})(\det F(z))^{\frac{1}{2}}\bar{P}_{\lambda,k,s}(-z)\right)_{L^2(\mathbb{R})}=$$

$$=\int_{\mathbb{R}}dz(\det F(z))P_{\lambda',k,s'}(-z)\tilde{f}(k,\lambda,z,s),$$

which does not make a precise sense. However, the trace of such elements,

$$\sum_{s=\pm1}\int_{Sp(\mathbf{\Delta}_h)}d\lambda\int_{\mathbb{R}}dz(\det F(z))P_{\lambda,k,s}(-z)\tilde{f}(k,\lambda,z,s),$$

can be given an exact sense if we change the order of integration,

$$\int_{\mathbb{R}}dz\sum_{s=\pm1}\int_{Sp(\mathbf{\Delta}_h)}d\lambda(\det F(z))P_{\lambda,k,s}(-z)\tilde{f}(k,\lambda,z,s).$$

Indeed, let $\{p_n(z)\}$ be an orthonormal system in $L^2(\mathbb{R})$. Consider their Fourier transforms $\tilde{p}_n(\lambda, k, s)$, and consider the following bi-distribution in the Fourier space, $\sum_{n=1}^{\infty} \overline{\widetilde{p_n}}(\lambda, k, s)\widetilde{p_n}(\lambda', k, s')$. Let $\tilde{f}, \tilde{g}$ be the Fourier transforms of arbitrary $f, g \in L^2(\mathbb{R})$. We have

$$\sum_{n=1}^{\infty} \sum_{s=\pm 1} \sum_{s'=\pm 1} \int_{Sp(\mathbf{\Delta}_h)} d\lambda \int_{Sp(\mathbf{\Delta}_h)} d\lambda' \overline{\widetilde{p_n}}(\lambda, k, s)\widetilde{p_n}(\lambda', k, s')\tilde{f}(\lambda, k, s)\bar{\tilde{g}}(\lambda', k, s') =$$

$$= \sum_{n=1}^{\infty} \left(\sum_{s=\pm 1} \int_{Sp(\mathbf{\Delta}_h)} d\lambda \overline{\widetilde{p_n}}(\lambda, k, s)\tilde{f}(\lambda, k, s) \right) \left(\sum_{s'=\pm 1} \int_{Sp(\mathbf{\Delta}_h)} d\lambda' \widetilde{p_n}(\lambda', k, s')\bar{\tilde{g}}(\lambda', k, s') \right) =$$

$$= \sum_{n=1}^{\infty} (p_n, f)_{L^2(\mathbb{R})} (g, p_n)_{L^2(\mathbb{R})} = (g, f)_{L^2(\mathbb{R})} = \sum_{s=\pm 1} \int_{Sp(\mathbf{\Delta}_h)} d\lambda \tilde{f}(\lambda, k, s)\bar{\tilde{g}}(\lambda, k, s),$$

thus $\sum_{n=1}^{\infty} \overline{\widetilde{p_n}}(\lambda, k, s)\widetilde{p_n}(\lambda', k, s') = \delta(\lambda - \lambda')\delta^s_{s'}$. Now

$$\int_{\mathbb{R}} dz \sum_{s=\pm 1} \int_{Sp(\mathbf{\Delta}_h)} d\lambda (\det F(z)) P_{\lambda,k,s}(-z)\tilde{f}(k, \lambda, z, s) =$$

$$= \int_{\mathbb{R}} dz (\det F(z)) \sum_{n=1}^{\infty} \sum_{s,s'} \iint_{Sp(\mathbf{\Delta}_h)^2} d\lambda d\lambda' \overline{\widetilde{p_n}}(\lambda, k, s)\widetilde{p_n}(\lambda', k, s') P_{\lambda,k,s}(-z)\tilde{f}(k, \lambda', z, s') =$$

$$= \int_{\mathbb{R}} dz (\det F(z)) \sum_{n=1}^{\infty} \overline{\left(\sum_{s=\pm 1} \int_{Sp(\mathbf{\Delta}_h)} d\lambda \widetilde{p_n}(\lambda, k, s)\bar{P}_{\lambda,k,s}(-z) \right)} \times$$

$$\times \left(\sum_{s'=\pm 1} \int_{Sp(\mathbf{\Delta}_h)} d\lambda' \tilde{f}(k, \lambda', z, s')\widetilde{p_n}(\lambda', k, s') \right) =$$

using (Eq.3.6),

$$= \int_{\mathbb{R}} dz (\det F(z)) \sum_{n=1}^{\infty} (\det F(z))^{-\frac{1}{2}} \overline{p_n}(z) (\det F(-z)) \hat{f}(T_{\check{k}_0(k)}) p_n(z) =$$

as both the sum and the integral converge in L^2,

$$= \sum_{n=1}^{\infty} \int_{\mathbb{R}} dz \overline{p_n}(z) (\det F(z))^{-\frac{1}{2}} \hat{f}(T_{\check{k}_0(k)}) p_n(z) = \sum_{n=1}^{\infty} \int_{\mathbb{R}} dz \overline{p_n}(z) D_\pi \hat{f}(T_{\check{k}_0(k)}) p_n(z) =$$

$$= \sum_{n=1}^{\infty} (p_n, D_\pi \hat{f}(T_{\check{k}_0(k)}) p_n)_{L^2(\mathbb{R})} = Tr\left[D_\pi \hat{f}(T_{\check{k}_0(k)}) \right].$$

Hence from (Eq.3.7) we have

$$f(1)=\int_{\mathfrak{K}} d\nu(k)\int_{\mathbb{R}} dz\sum_{s=\pm1}\int_{Sp(\mathbf{\Delta}_h)} d\lambda(\det F(z))P_{\lambda,k,s}(-z)\tilde{f}(k,\lambda,z,s).$$

To find an inversion formula at arbitrary point $g\in G$ we apply this to the left translated function $[L_{g^{-1}}f](x)=f(gx)$,

$$f(g)=[L_{g^{-1}}f](1)=\int_{\mathfrak{K}} d\nu(k)\int_{\mathbb{R}} dz\sum_{s=\pm1}\int_{Sp(\mathbf{\Delta}_h)} d\lambda(\det F(z))P_{\lambda,k,s}(-z)[\widetilde{L_{g^{-1}}f}](\lambda,k,z,s).$$

But from the definition

$$[\widetilde{L_{g^{-1}}f}](\lambda,k,r,s)=\int_G dh\bar{\zeta}_{k,\lambda,r,s}(h)[L_{g^{-1}}f](h)=\int_G dh'\bar{\zeta}_{k,\lambda,r,s}(g^{-1}h')f(h').$$

From the definition of $\zeta_{k,\lambda,r,s}$ we find

$$\bar{\zeta}_{k,\lambda,r,s}(g^{-1}h')=e^{-i(F^{\perp}(r+g_z)\check{k}_0(k),\check{g})}(\det F(g_z))\bar{\zeta}_{\lambda,k,r+g_z,s}(h'),$$

thus

$$\int_G dh'\bar{\zeta}_{k,\lambda,r,s}(g^{-1}h')f(h')=e^{-i(F^{\perp}(r+g_z)\check{k}_0(k),\check{g})}(\det F(g_z))\tilde{f}(k,\lambda,r+g_z,s).$$

Therefore

$$f(g)=\int_{\mathfrak{K}} d\nu(k)\int_{\mathbb{R}} dz\sum_{s=\pm1}\int_{Sp(\mathbf{\Delta}_h)} d\lambda(\det F(z))P_{\lambda,k,s}(-z)\times$$

$$\times e^{-i(F^{\perp}(z+g_z)\check{k}_0(k),\check{g})}(\det F(g_z))\tilde{f}(k,\lambda,z+g_z,s)=$$

by substitution $r=z+g_z$

$$=\int_{\mathfrak{K}} d\nu(k)\int_{\mathbb{R}} dr\sum_{s=\pm1}\int_{Sp(\mathbf{\Delta}_h)} d\lambda(\det F(r))\tilde{f}(k,\lambda,r,s)\times$$

$$\times e^{-i(F^{\perp}(r)\check{k}_0(k),\check{g})}P_{\lambda,k,s}(g_z-r)=$$

$$=\int_{\mathfrak{K}} d\nu(k)\int_{\mathbb{R}} dr\sum_{s=\pm1}\int_{Sp(\mathbf{\Delta}_h)} d\lambda(\det F(r))\tilde{f}(k,\lambda,r,s)\zeta_{k,\lambda,r,s}(g),$$

which is our final inversion formula.

It remains to note that by denoting $\alpha=(k,\lambda,r,s)$ we have satisfied all conditions for the eigenfunction expansion $\bar{\zeta}_\alpha(f)$ to give a conventional Fourier transform.

3.8 Harmonic analysis on the Bianchi VIII group

The Bianchi VIII group is the universal covering group $\widetilde{SL(2,\mathbb{R})}$ of the semisimple Lie group $SL(2,\mathbb{R})$ whose fundamental group is $\mathbb{Z}$. Therefore the center of $\widetilde{SL(2,\mathbb{R})}$ is $\mathbb{Z}$, i.e., infinite, which makes the harmonic analysis difficult (an essential part of Helgason's theory applies for semisimple Lie groups with finite center). However it has been done, and we present here briefly the main results established in [47].

Consider the covering homomorphism $\widetilde{SL(2,\mathbb{R})} \xrightarrow{\Phi} SL(2,\mathbb{R})$. In (Pukanszky) there was choosen a global chart $(\lambda,\mu,\phi) \in \mathbb{R}_+ \times \mathbb{R}^2$ on $\widetilde{SL(2,\mathbb{R})}$ such that Φ simply identifies ϕ with $\phi + 2\pi\mathbb{Z}$. This allows to find the group operations of $\widetilde{SL(2,\mathbb{R})}$ explicitly from those of $SL(2,\mathbb{R})$ which are by matrices. The resulting expressions are however very cumbersome, and leave little hope for explicit constructions. The eigenfunctions of the Laplacian on $SL(2,\mathbb{R})$ could be lifted to yield some of the eigenfunctions on $\widetilde{SL(2,\mathbb{R})}$, but these will constitute only a tiny minority. The problem of finding the complete system of eigenfunctions ζ_α remains fairly hypothetic.

According [47], a Plancherel essential part of $\hat{G}$ (we call it $\tilde{K}$) consists of three families of representations: $C_\sigma^{(\tau)}$ $(0 \le \tau < 1,\ \sigma > 0)$, D_l^+ and D_l^- $(l > \frac{1}{2})$. As we see, $\tilde{K}$ is a smooth manifold consisting of three connected components. If we knew that the spectrum of Δ_h consists of contiguous intervals of $\mathbb{R}$, then we could already construct a conventional Fourier transform. Unfortunately we will not have the opportunity to do this here. The Plancherel formula has the following form,

$$f(1) = \int_0^\infty \int_0^1 d\tau d\sigma \sigma \Re\left[\tanh \pi(\sigma + i\tau)\right] Tr[C_\sigma^{(\tau)}(f)] + \int_{\frac{1}{2}}^\infty dl(l - \frac{1}{2}) \left(Tr[D_l^+(f)] + Tr[D_l^-(f)]\right).$$

3.9 Harmonic and Fourier analysis on the Bianchi IX group

The Bianchi IX class is probably the luckiest in enjoying the attention of the scientific society, because it is related to maybe the most fundamental groups in physics, $SO(3)$ and $SU(2)$. The Lie algebras $so(3)$ and $su(2)$ are isomorphic, and the connected simply connected Lie group with this algebra is $SU(2)$, which is the universal covering group of $SO(3)$. Hence we will concentrate on $SU(2)$ endowed with any left invariant Riemannian metric. The harmonic analysis of $SU(2)$ is concisely described, for instance, in [21], and we summarize it here.

The group $SU(2)$ can be given as a subgroup of matrices of the form

$$U_{a,b} = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix},$$

more precisely, $SU(2) = \{U_{a,b}\colon\ a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1\}$. The correspondence $U_{a,b} \to (a, b)$ identifies $SU(2)$ with the unit sphere $S^3 \subset \mathbb{C}^2$, where the identity element corresponds to the north pole $(1, 0)$. Thus one can consider functions on $SU(2)$ twofold: either as functions of the intrinsic angle coordinates (e.g., (5.29)in [21]), or as the restrictions to S^3 of functions on $\mathbb{C}^2 \sim \mathbb{R}^4$. The matrix multiplication law $g \cdot f = h$ considered as the left action of $SU(2)$ on $\mathbb{C}^2$ is given by

$$U_{a,b}(z, w) = (az - \bar{b}w, bz + \bar{a}w),\ (z, w) \in \mathbb{C}^2.$$

Consider the linear space $\mathcal{P}_m$ of homogeneous polynomials of degree m on $\mathbb{C}^2$,

$$\mathcal{P}_m = \{P\colon\ P(z, w) = \sum_0^m c_j z^j w^{m-j},\ c_j \in \mathbb{C}\}.$$

Let $d\sigma$ be the Lebesgue surface measure on S^3 normalized by $\sigma(S^3) = 1$, and consider the $L^2(S^3)$ product on $\mathcal{P}_m$,

$$(P, Q) = \int_{S^3} d\sigma \bar{P} Q.$$

$\mathcal{P}_m$ becomes a finite dimensional Hilbert space through completion under this inner product. For a function $f \in C(\mathbb{C}^2)$ the left action of $SU(2)$ is given by

$$\pi(U_{a,b}) f(z, w) = f(U_{a,b}^{-1}(z, w)) = f(\bar{a}z + \bar{b}w, -bz + aw).$$

Denote by π_m the restriction of this action to each $\mathcal{P}_m$. It turns out, that all π_m, $m \geq 0$ are unitary and irreducible, and moreover, $(\widehat{SU(2)}) = \{[\pi_m],\ m \geq 0\}$ (see [21]). The Fourier transform is given by (here we again change the original definition to have a common form with non unimodular groups)

$$\hat{f}(m) = \hat{f}(\pi_m) = \int f(x) \pi(x) dx,$$

and the inversion formula by

$$f(x) = \sum_{0}^{\infty}(m+1)Tr\left[\pi^*(x)\hat{f}(m)\right].$$

The standard Laplacian on $\mathbb{C}^4$,

$$\Delta = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_4^2} = 4\frac{\partial^2}{\partial a\partial\bar{a}} + 4\frac{\partial^2}{\partial b\partial\bar{b}},\ (x_1,...,x_4)\in\mathbb{R}^4,\ (a,b)\in\mathbb{C}^2,$$

is left and right invariant under the action of $SU(2)$. We have for $N=4$-dimensional Laplacian the following polar form,

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^{N-1}} = \frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^3},$$

where Δ_{S^3} is the spherical Laplacian on S^3. Because the action of $SU(2)$ conserves r, from the left and right invariance of Δ follows the left and right invariance of Δ_{S^3}, i.e., Δ_{S^3} is Casimir. The matrix elements $\pi_m^{jk}(a,b)$ are the restrictions to S^3 of polynomials in $\mathbb{C}^2$ of homogeneous degree m in the variable r, and can be written as

$$\pi_m^{jk}(a,b) = r^m\pi_m^{jk}(\theta),\ \theta\in S^3.$$

It can be seen (Folland), that $\Delta\pi_m^{jk}(a,b)=0$, i.e.,

$$\Delta\pi_m^{jk}(a,b) = \left(\frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^3}\right)r^m\pi_m^{jk}(\theta) = r^{m-2}(m(m+2)+\Delta_{S^3})\pi_m^{jk}(\theta) = 0,$$

hence

$$\Delta_{S^3}\pi_m^{jk}(\theta) = -m(m+2)\pi_m^{jk}(\theta).$$

The eigenfunctions $\pi_m^{jk}(a,b)$ are the spherical harmonics given by

$$\pi_m^{jk}(a,b) = \sqrt{\frac{j!(m-j)!}{k!(m-k)!}}\int_0^1(\bar{a}e^{2\pi it}+\bar{b})^k(-be^{2\pi it}+a)^{m-k}e^{-2\pi ijt}dt.$$

For a convenient enumeration and relation to Jacobi polynomials refer, e.g., to [59].

Now let (X_1,X_2,X_3) be the standard basis of the left invariant fields on $SU(2)$ with commutation relations $[X_i,X_j]=\varepsilon_{ij}^kX_k$, where ε_{ij}^k is the totally anti-symmetric matrix. Let further $\Delta_h=\sum_{i,j=1}^3\hat{h}^{ij}X_iX_j$ be the metric Laplacian with the positive real symmetric matrix $\hat{h}^{ij}$. We write

$$[X_k, \Delta_h] = \sum_{i,j=1}^{3} \hat{h}^{ij}([X_k, X_i]X_j + X_i[X_k, X_j]) = \sum_{i,j,l=1}^{3} \varepsilon^l_{kj}\hat{h}^{ji}(X_l X_i + X_i X_l) \neq 0,$$

and see that Δ_h is not Casimir in general. The determination of the explicit form of its eigenfunctions seems to be a hard task. In particular, rather massive but elementary calculations show that neither a $\mathbb{C}^2$-linear nor even an $\mathbb{R}^4$-linear transformation can in general diagonalize $\hat{h}^{ij}$ while preserving the Lie algebra structure. Thus the eigenfunnctions of Δ_h cannot be reduced to spherical harmonics by merely a linear transformation. Unfortunately we have to leave this problem open. If $\hat{h} = 1$ then the Riemannian structure has higher symmetry, namely, it is an FRW spacetime. FRW spacetimes will be considered separately, so for general Bianchi IX spacetimes we do not know the eigenfunctions ζ_α explicitly.

3.10 Harmonic and Fourier analysis on FRW spaces

By FRW spaces we mean homogeneous spaces G/O which model the spatial sections of FRW spacetimes. These are semidirect homogeneous spaces, $G = M \rtimes O$, for three maximal isometry groups: $SO(4) = SU(2) \rtimes SO(3)$, $E(3) = \mathbb{R}^3 \rtimes SO(3)$ and $SO^+(1,3) = Bi(V) \rtimes SO(3)$ ($Bi(V)$ is the Bianchi V group), which are all unimodular groups. The left quasi-regular representations U_g for all three cases are well studied in the literature, and we will merely state the known facts.

Let us start with $G = SO(4)$. The dual space $\hat{G}_M = \mathbb{N}_0$, each representation π_k is $(k+1)^2$-dimensional, and the multiplicity $mult(\pi_k, U_g) = 1$. The Laplace operator Δ acts therefore as a scalar in each $\mathcal{H}(\pi_k)$, and that scalar equals $\lambda = -k(k+2)$. The eigenfunctions ζ_α are the spherical harmonics, where $\tilde{\Sigma} \ni \alpha = (k, l, m)$, $0 \leq l \leq k$ and $m^2 \leq l$, $l, m \in \mathbb{Z}$.

Next consider $G = E(3)$. The dual space $\hat{G}_M = \mathbb{R}_+$, all representations are infinite dimensional, but the multiplicities are again $mult(\pi_k, U_g) = 1$. The corresponding eigenvalue of Δ is $\lambda = -k^2$. The Fourier space can be modelled by $\alpha \in \tilde{\Sigma} = \mathbb{R}^3$ such that the sphere $|\alpha|^2 = k^2$ corresponds to the representation π_k. The eigenfunctions are $\zeta_\alpha(x) = (2\pi)^{3/2}e^{i(\alpha,x)}$, where $(,)$ here is the Euclidean product. The spectral measure is

$d\mu(\pi_k) = k^2 dk.$

Finally let $G = SO^+(1,3)$. This homogeneous space is called Lobachevsky space. The dual space is again $\hat{G}_M = \mathbb{R}_+$, each representation being infinite dimensional with multiplicity $mult(\pi_k, U_g) = 1$. The eigenvalues of the Laplace operator are $\lambda = -(k^2+1)$. The harmonic analysis by means of integral geometry on Lobachevsky space can be found, for instance, in [24] and [59]. The Fourier space $\tilde{\Sigma}$ can be modelled by $\tilde{\Sigma} = \{\alpha = (k, \vec{\eta})\} = \mathbb{R}_+ \times S^2$, and the eigenfunctions are $\zeta_\alpha(x) = (2\pi)^{-\frac{3}{2}}(\vec{x}, \vec{\eta})^{ik-1}$, where M is considered being embedded in $\mathbb{R}^4$ as $M = \{x \in \mathbb{R}^4 : [x,x] = 1\}$, so that $x = (x_0, \vec{x})$ and $\eta = (1, \vec{\eta})$, $[\eta, \eta] = 0$ ($[,]$ is the Minkowski scalar product). The disadvantage of this Fourier transform is that $\bar{\zeta}_{(k,\vec{\eta})} = \zeta_{(-k,\vec{\eta})}$ does not enter (k and $-k$ give equivalent representations, as we have seen from abstract considerations). To obtain a conventional Fourier transform we suggest to extend $\tilde{\Sigma}$ to include also $k < 0$, i.e., $\tilde{\Sigma} = \mathbb{R} \times S^2$. Then to retain the Plancherel formula we have to renormalize the eigenfunctions, $\zeta_\alpha(x) = 4^{-1}(\pi)^{-\frac{3}{2}}[\vec{x}, \vec{\eta}]^{ik-1}$. The price we have to pay is that now the Fourier image of $L^2(M)$ is not $L^2(\tilde{\Sigma}, d\mu)$, but only those $\tilde{f} \in L^2(\tilde{\Sigma}, d\mu)$ which satisfy

$$\int_{S^2} d\vec{\eta} \tilde{f}(\vec{\eta}, \rho) \zeta_{(\vec{\eta},\rho)}(x) = \int_{S^2} d\vec{\eta} \tilde{f}(\vec{\eta}, -\rho) \zeta_{(\vec{\eta},\rho)}(x), \ \forall x \in M.$$

The spectral measure is $d\mu(\pi_k) = k^2 dk$. For details we refer to the above mentioned books.

3.11 Automorphism groups of Bianchi I-VII groups

In this section we consider the automorphism groups Aut(G) of Bianchi I-VII groups. After performing the calculations we discovered that these automorphisms have been obtained earlier in [27]. However we give here also the dual actions of these automorphisms on $\hat{G}$ which is new. This may become important when analyzing the transformation in the Fourier space induced by automorphisms. We start by noting that Bianchi I-VII groups are matrix groups, and their matrix realization can be given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow G(x,y,z) = \begin{pmatrix} F(z) & & x \\ & & y \\ 0 & 0 & 1 \end{pmatrix}.$$

It can be easily seen that in this realization the group multiplication indeed corresponds to the matrix multiplication. The respective Lie algebra realization will be

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \mathfrak{g}(x,y,z) = \begin{pmatrix} zM & & x \\ & & y \\ 0 & 0 & 0 \end{pmatrix},$$

which again can be checked to intertwine the matrix commutation with the Lie bracket. Moreover, we could have obtained immediately the exponential map by setting $\exp(x,y,z) = \exp(\mathfrak{g}(x,y,z))$ instead of referring to the Zassenhaus formula, but the latter is a more Lie theoretical approach. Now that all Bianchi groups are connected and simply connected by Theorem 1 of III.6.1 in [5] it follows $\mathrm{Aut}(G) = \mathrm{Aut}(\mathfrak{g})$ in sense of a topological group isomorphism (see also [31]). An algebra homomorphism of matrix algebras is necessarily linear in the matrix elements. It follows that any $\breve{\alpha} \in \mathrm{Aut}(\mathfrak{g})$ depends linearly on x, y, z, and is therefore given by an affine transformation in $\mathbb{R}^3$, which is actually a linear transformation because it preserves 0. Therefore we first determine $\mathrm{Aut}(\mathfrak{g})$. Let the linear map $\breve{\alpha} : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \breve{\alpha}_{2\times 2} & \breve{\alpha}_{\cdot 3} \\ \breve{\alpha}_{3\cdot} & \breve{\alpha}_{33} \end{pmatrix} \begin{pmatrix} q \\ r \\ s \end{pmatrix}.$$

Then $\breve{\alpha} \in \mathrm{Aut}(\mathfrak{g})$ if and only if $\breve{\alpha}[\vec{x}, \vec{y}] = [\breve{\alpha}\vec{x}, \breve{\alpha}\vec{y}]$, where $[,]$ is the Lie bracket. Expending this condition we get the system of requirements

$$\breve{\alpha}_{2\times 2}M - \breve{\alpha}_{33}M\breve{\alpha}_{2\times 2} + M\breve{\alpha}_{\cdot 3}\breve{\alpha}_{3\cdot} = 0, \tag{3.8}$$

$$\breve{\alpha}_{2\times 2}M\sigma\breve{\alpha}_{3\cdot}^{\top} = 0,$$

$$\breve{\alpha}_{3\cdot}M = 0,$$

where σ is the unit antisymmetric matrix. The patterns of admissible matrices $\breve{\alpha}$ satisfying this system have to be computed for each group independently. For Bianchi I we have $M = 0$ and all three conditions are satisfied trivially. For Bianchi IV-VII the matrix M is invertible hence the third requirement means $\breve{\alpha}_{3\cdot} = 0$, so that the second becomes trivial, and the first reduces to $\breve{\alpha}_{2\times 2}M - \breve{\alpha}_{33}M\breve{\alpha}_{2\times 2} = 0$. The cases of groups Bianchi II and III are a bit more involved, but the calculations are straightforward. We present the results in the Table 3.11. Note that whenever $\breve{\alpha}_{3\cdot} = 0$ the invertibility of $\breve{\alpha}$ requires $\breve{\alpha}_{33} \neq 0$.

I	II	III	IV	V	VI, $q \neq 1$
$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$	$\begin{pmatrix} a & 0 & c \\ d & a \cdot i - c \cdot g & f \\ g & 0 & i \end{pmatrix}$	$\begin{pmatrix} a & 0 & c \\ 0 & e & f \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} a & 0 & c \\ d & a & f \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} a & 0 & c \\ 0 & e & f \\ 0 & 0 & 1 \end{pmatrix}$

VI, $q = 1$	VII, $p \neq 0$	VII, $p = 0$
$\begin{pmatrix} a & 0 & c \\ 0 & e & f \\ 0 & 0 & 1 \end{pmatrix} \bigcup \begin{pmatrix} 0 & b & c \\ d & 0 & f \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} a & b & c \\ -b & a & f \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} a & b & c \\ -b & a & f \\ 0 & 0 & 1 \end{pmatrix} \bigcup \begin{pmatrix} a & b & c \\ b & -a & f \\ 0 & 0 & -1 \end{pmatrix}$

Table 3.3: Patterns of permissible matrices $\breve{\alpha}$ for Bianchi I-VII algebras

As it can be seen from the table some algebras allow for reflective automorphisms and their automorphism groups consist of two components. Matrices of these pattern forms exhaust the groups $\mathrm{Aut}(\mathfrak{g})$. One can compare this patterns of automorphisms to those available in the literature, for instance, of the Heisenberg algebra in (Folland.HAPS).

Now the corresponding group homomorphisms $\breve{A} \in \mathrm{Aut}(G)$ can be found by composing $\breve{\alpha} \in \mathrm{Aut}(\mathfrak{g})$ with the exponential map, $\breve{A}\exp((x,y,z)) = \exp(\breve{\alpha}(x,y,z))$. Recall that the exponential map is given by

$$\exp((x,y,z)) = ([1 + F(z)D(z)](x,y),z),$$

and because this map is bijective we know that the matrix $[1 + F(z)D(z)]$ is invertible for all z. The logarithmic map can be written as

$$\log((x,y,z)) = ([1 + F(z)D(z)]^{-1}(x,y),z),$$

and the action of the group homomorphism $\breve{A}$ related to the algebra homomorphism $\breve{\alpha}$ becomes

$$\breve{A}\begin{pmatrix} \breve{x} \\ z \end{pmatrix} = \begin{pmatrix} [1 + F(z')D(z')]\,(\breve{\alpha}_{2\times 2}[1 + F(z)D(z)]^{-1}\breve{x} + \breve{\alpha}_{.3}z) \\ z' \end{pmatrix}$$

$$z' = \breve{\alpha}_{3.}[1 + F(z)D(z)]^{-1}\breve{x} + \breve{\alpha}_{33}z,$$

for Bianchi II-VII groups and

$$\breve{A}\begin{pmatrix} \breve{x} \\ z \end{pmatrix} = \breve{\alpha}\begin{pmatrix} \breve{x} \\ z \end{pmatrix}$$

for Bianchi I group. From $\breve{\alpha}_{3.}M = 0$ it follows $\breve{\alpha}_{3.}[1 + F(z)D(z)]^{-1} = \breve{\alpha}_{3.}$ thus the formula for Bianchi II-VII simplifies to

$$\check{A}\begin{pmatrix}\check{x}\\ z\end{pmatrix} = \begin{pmatrix}[1+F(\check{\alpha}_{3.}\check{x}+\check{\alpha}_{33}z)D(\check{\alpha}_{3.}\check{x}+\check{\alpha}_{33}z)]\,(\check{\alpha}_{2\times 2}[1+F(z)D(z)]^{-1}\check{x}+\check{\alpha}_{.3}z)\\ \check{\alpha}_{3.}\check{x}+\check{\alpha}_{33}z\end{pmatrix}.$$

One more step can be done in this generality. From (Eq.3.8) and $\check{\alpha}_{3.}M = 0$ it follows

$$\check{\alpha}_{2\times 2}M^m = (\check{\alpha}_{33}M)^m\check{\alpha}_{2\times 2}$$

for $m \geq 2$ and therefore

$$\check{\alpha}_{2\times 2}\sum_{m=0}^{\infty} f_m M^m = \sum_{m=0}^{\infty} f_m(\check{\alpha}_{33}M)^m\check{\alpha}_{2\times 2} + f_1 M\check{\alpha}_{.3}\check{\alpha}_{3.}$$

whenever the left side exists. This can be used to establish that

$$[1+F(\check{\alpha}_{3.}\check{x}+\check{\alpha}_{33}z)D(\check{\alpha}_{3.}\check{x}+\check{\alpha}_{33}z)]\check{\alpha}_{2\times 2} = \check{\alpha}_{2\times 2}[1+F(\frac{\check{\alpha}_{3.}\check{x}}{\check{\alpha}_{33}}+z)D(\frac{\check{\alpha}_{3.}\check{x}}{\check{\alpha}_{33}}+z)].$$

This far on the explicit form of the group automorphisms.

Now let us look at the dual spaces $\hat{G}$. If $\check{A} \in \mathrm{Aut}(G)$ and $\pi \in \hat{G}$ then $\pi \circ \check{A} = \pi'$ for some $\pi' \in \hat{G}$. Thus $\check{A}$ induces a pullback map $\check{A}^* : \hat{G} \to \hat{G}$. Because $\dim \pi = \dim \pi'$ it follows that $\check{A}^*$ maps generic representations into generic representations and singletons into singletons. The representations $\pi \in \hat{G}$ are in a bijective correspondence with the derived representations $d\pi$ which are irreducible representations of the Lie algebra $\mathfrak{g}$. In a similar fashion, any $\check{\alpha} \in \mathrm{Aut}(\mathfrak{g})$ induces a pullback map $\alpha^* : d\hat{G} \to d\hat{G}$ between derived representations. This pullback map is easier to study than that for the group representations. Consider first the Bianchi I group. The irreducibles are given by

$$T_{\vec{k}}(\vec{g}) = e^{i(\vec{k},\vec{g})},$$

and the derived representations are

$$dT_{\vec{k}}(\vec{x}) = i(\vec{k},\vec{x}).$$

An automorphism $\vec{x} = \check{\alpha}\vec{q}$ induces the pullback map $\check{\alpha}^*(\vec{k}) = \check{\alpha}^\top\vec{k}$. Consider now the singletons of a Bianchi II-VII group. They are given for $\vec{k} \in V^0 \oplus \mathbb{R}$ by

$$T_{\vec{k}}(\vec{g}) = e^{i(\vec{k},\vec{g})} = e^{i(\check{k},\check{g})}e^{ik_3 g_z},$$

and the derived singletons are

$$dT_{\vec{k}}(\vec{x}) = i(\vec{k}, \vec{x}),$$

and again, an automorphism $\vec{x} = \breve{\alpha}\vec{q}$ induces the pullback map $\breve{\alpha}^*(\vec{k}) = \breve{\alpha}^\top \vec{k}$. This in particular means that $\breve{k}' = \breve{\alpha}_{2\times 2}^\top \breve{k} + k_3 \breve{\alpha}_{3\cdot}^\top$, and if $\breve{k} \in V^0$ then

$$M^\top \breve{k}' = M^\top \breve{\alpha}_{2\times 2}^\top \breve{k} + k_3 M^\top \breve{\alpha}_{3\cdot}^\top = 0,$$

where (Eq.3.8) and $\breve{\alpha}_{3\cdot} M = 0$ were used. We explicitly observe that the automorphisms map singletons into singletons, as expected. Finally we turn to the generic representations. Let $T_{\breve{k}}$ be a generic representation of G. Then it acts on $L^2(\mathcal{R})$ by

$$T_{\breve{k}}(\vec{g}) f[w] = e^{i(\breve{k}, F(-w)\breve{g})} f[w - g_z], \; \vec{g} = (\breve{g}, g_z) \in G.$$

Its derived representation will be

$$dT_{\breve{k}}(\vec{x}) f[w] = i(\breve{k}, F(-w)\breve{x}) f[w] - z\partial_w f[w].$$

Under the automorphism $\vec{x} = \breve{\alpha}\vec{q}$ it will turn into

$$dT_{\breve{k}}(\vec{q}) f[w] = i(\breve{k}, F(-w)[\breve{\alpha}_{2\times 2}\breve{q} + \breve{\alpha}_{\cdot 3} s]) f[w] - [\breve{\alpha}_{3\cdot}\breve{q} + \breve{\alpha}_{33} s]\partial_w f[w].$$

For simplicity we will consider only the automorphisms with $\breve{\alpha}_{3\cdot} = 0$. Thus we omit only some automorphisms of the Heisenberg group, but this group is a central subject in the harmonic analysis, and the missing results can be probably found in the literature. Define the isometric isomorphism $\mathfrak{T} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$\mathfrak{T}(f)[w] = \frac{1}{\sqrt{\breve{\alpha}_{33}}} e^{i(\breve{k}, \int_0^{-w} F(\breve{\alpha}_{33}\xi) d\xi \breve{\alpha}_{\cdot 3})} f(\breve{\alpha}_{33} w).$$

Consider the representation $dT_{\breve{k}'}$ with $\breve{k}' = \breve{\alpha}_{2\times 2}^\top \breve{k}$. Note that because $\breve{\alpha}_{3\cdot} = 0$ we have that $\breve{\alpha}_{2\times 2}^\top$ is invertible, and from (Eq.3.8) we assure that it maps $\breve{k} \notin V^0$ to $\breve{k}' \notin V^0$. Thus $dT_{\breve{k}'}$ is generic. Its action on the image $\mathfrak{T}(f)[w]$ is given by

$$dT_{\breve{k}'}(\vec{q}) \mathfrak{T}(f)[w] = i(\breve{k}', F(-w)\breve{q}) \mathfrak{T}(f)[w] - s\partial_w \mathfrak{T}(f)[w] =$$

$$= \mathfrak{T}(i(\breve{k}', F(-\frac{\bullet}{\breve{\alpha}_{33}})\breve{q}) f)[w] + \mathfrak{T}(is(\breve{k}, F(-\bullet)\breve{\alpha}_{\cdot 3}) f)[w] - \mathfrak{T}(s\breve{\alpha}_{33}\partial f).$$

Recall that we have seen that from (Eq.3.8) and $\breve{\alpha}_{3\cdot} = 0$ it follows $\breve{\alpha}_{2\times 2} F(z) = F(\breve{\alpha}_{33} z)\breve{\alpha}_{2\times 2}$, hence

$$i(\breve{k}', F(-\frac{\bullet}{\breve{\alpha}_{33}})\breve{q}) = i(\breve{\alpha}_{2\times 2}^\top \breve{k}, F(-\frac{\bullet}{\breve{\alpha}_{33}})\breve{q}) = i(\breve{k}, F(-\bullet)\breve{\alpha}_{2\times 2}\breve{q}).$$

We finally see that

$$dT_{\breve{k}'}(\vec{q})\mathfrak{T}(f)[w] = \mathfrak{T}\left([i(\breve{k}, F(-\bullet)\breve{\alpha}_{2\times 2}\breve{q} + is(\breve{k}, F(-\bullet)\breve{\alpha}_{\cdot 3})]f - s\breve{\alpha}_{33}\partial f\right) = \mathfrak{T}\left(dT_{\breve{k}}(\vec{q})f\right),$$

which means that $\mathfrak{T}$ intertwines the irreducible representations $dT_{\breve{k}} \circ \breve{\alpha}$ and $dT_{\breve{\alpha}_{2\times 2}^{\top}\breve{k}}$. Thus these two representations are unitarily equivalent, $\breve{\alpha}^*(\breve{k}) = \breve{\alpha}_{2\times 2}^{\top}\breve{k}$. If the cross sections are chosen explicitly (for instance, as we did) then it is a straightforward calculation to find the action of $\breve{\alpha}^*$ on $\tilde{K}$ and $\mathfrak{K}$. We omit these calculations here because, first, they depend on the preferred choice of the cross sections, and second, they involve transcendental functions (e.g., the solution of the equation $e^y + ay = x$) and are not transparent visually, and do not provide a better insight on the matter.

3.12 Separation of time variable in homogeneous universes

We want to see to which extent the technique of mode decomposition is applicable to the Bianchi type and FRW cosmological models. For this aim we have to check whether the conditions of **Proposition 1.3** are satisfied. Recall that the metric of a homogeneous spacetime is given by

$$ds^2 = dt^2 - \sum_{\alpha,\beta=1}^{3} \breve{h}_{\alpha\beta}(t) d\omega^\alpha(\vec{x}) d\omega^\beta(\vec{x}),$$

where $\breve{h}_{ij}(t)$ is a smooth positive symmetric matrix function, and $d\omega^i$ are the left invariant 1-forms on Σ_t. The condition (i) of **Proposition 1.3** is automatically satisfied because $g_{00} = 1$. For the condition (ii) note that

$$\sum_{i,j=1}^{3} g^{ij}(x)\frac{\partial g_{ij}}{\partial t}(x) = \sum_{i,j=1}^{3}\sum_{\alpha,\beta=1}^{3}\sum_{\gamma,\delta=1}^{3} \breve{h}^{\alpha\beta}(t)\dot{\breve{h}}_{\gamma\delta}(t) X^i_\alpha(\vec{x}) X^j_\beta(\vec{x}) (d\omega^\gamma)_i(\vec{x})(d\omega^\delta)_j(\vec{x}) =$$

$$= \sum_{\alpha,\beta=1}^{3}\sum_{\gamma,\delta=1}^{3} \breve{h}^{\alpha\beta}(t)\dot{\breve{h}}_{\gamma\delta}(t)\langle X_\alpha, d\omega^\gamma\rangle_h \langle X_\beta, d\omega^\delta\rangle_h = \sum_{\alpha,\beta=1}^{3}\sum_{\gamma,\delta=1}^{3} \breve{h}^{\alpha\beta}(t)\dot{\breve{h}}_{\gamma\delta}(t)\delta^\gamma_\alpha\delta^\delta_\beta = Tr[\breve{h}^{-1}(t)\dot{\breve{h}}(t)].$$

This shows that the condition (ii) is also satisfied. We see that the homogeneous spacetimes are an ideal playground for the mode decomposition. Note that FRW spacetimes correspond to the choice $\check{h}_{\alpha\beta}(t) = a^2(t)\delta_{\alpha\beta}$.

The satisfaction of the conditions (iii) and (iv) depend on the chosen connection ∇. For the scalar field (iii) is automatically satisfied with $\Gamma = 0$. The condition (iv) can be obviously satisfied if $\check{h}_{\alpha\beta}(t) = a^2(t)\check{h}^0_{\alpha\beta}$ as it amounts only to a rescaling of $\lambda(\alpha)$ in $D_{\Sigma_t}\zeta_\alpha = \lambda(\alpha)\zeta_\alpha$. Note that because D_{Σ_t} is G-invariant, the term m^2 is a function of t only. This is the situation where the dynamics of the universe consists of merely an isotropic rescaling. Thus, for instance, in case of FRW spacetimes the condition (iv) is satisfied automatically.

But the condition (iv) can be also satisfied non-trivially with an anistropic rescaling and even some shears and rotations. This is clearly possible for Bianchi I group, because the eigenfunctions do not depend on the matrix $\check{h}$. For Bianchi II-VII groups one has to look at the equation (Eq.3.4) to see to which extent the solution $P(z)$ depends on the matrix $\check{h}$. Suppose $\check{h}$ and $\check{j}$ are two matrices for which there exist two common linearly independent solutions $P(z)$ and $Q(z)$. Because we have already seen that an isotropic rescaling is always possible, without loss of generality we assume $\check{h}^{33} = \check{j}^{33}$. Now the condition that the two equations have the same solution spaces can be cast into the following pair of equations,

$$\check{h}^{3\cdot}F^\top(z)\check{k} = \check{j}^{3\cdot}F^\top(z)\check{q},$$

$$\lambda + \check{k}^\top F(z)\check{h}^{2\times 2}F^\top(z)\check{k} = \lambda' + \check{q}^\top F(z)\check{j}^{2\times 2}F^\top(z)\check{q}$$

for some $\check{q}$, λ' and for all $z \in \mathbb{R}$. That non-trivial possibilities exist is clear visually, but we will not go into details here. Once this conditions are satisfied for the 1-parameter family of matrices $\check{h}(t)$ then the condition (iv) is satisfied, and we have an explicit formula for the time dependent eigenvalue $\lambda(t)$.

As the electromagnetism is of primary importance for us, let us finally show that the assumption $\check{h}_{\alpha\beta}(t) = a^2(t)\check{h}^0_{\alpha\beta}$ is sufficient to satisfy the condition (iii) for the 1-form field. Indeed, the 1-form field is given by the Levi-Civita connection, for which the connection forms are $(\Gamma_i)^a_b = -\mathbf{\Gamma}^a_{ib}$. Let us compute the symbol $\mathbf{\Gamma}^a_{0b}$. It is easy to see that $\mathbf{\Gamma}^0_{0b} = \mathbf{\Gamma}^a_{00} = 0$. For $a, b > 0$ we have

$$\mathbf{\Gamma}^a_{0b} = \frac{1}{2}\sum_{m=1}^{3} g^{am}\frac{\partial g_{mb}}{\partial t} = \sum_{\alpha,\beta,\gamma=1}^{3} \check{h}^{\alpha\beta}(t)\dot{\check{h}}_{\beta\gamma}(t)X^a_\alpha(\vec{x})d\omega^\gamma_b(\vec{x}).$$

If $\check{h}_{\alpha\beta}(t) = a^2(t)\check{h}^0_{\alpha\beta}$ then $\dot{\check{h}}_{\alpha\beta}(t) = 2H(t)\check{h}_{\alpha\beta}(t)$, and we get

$$\mathbf{\Gamma}^a_{0b} = 2H(t) \sum_{\alpha,\beta,\gamma=1}^{3} \check{h}^{\alpha\beta}(t)\check{h}_{\beta\gamma}(t)X^a_\alpha(\vec{x})d\omega^\gamma_b(\vec{x}) = 2H(t)\delta^a_b.$$

Thus $\Gamma_0 = -2H(t)0 \oplus \mathbf{1}_3$ is not only a function of t, but also commutes with any matrix, hence (iii) is trivially satisfied.

Chapter 4

Mode decomposition of quantum fields

4.1 Locally covariant quantum fields

In this chapter we start dealing with quantum fields. It will be shown how the apparatus developed for the classical fields in previous chapters can be applied to several problems on the quantum level. In particular, the 2-point functions of quasifree states will be analyzed as weak bi-solutions of the field equation.

In the current section we will define locally covariant quantum field theories, and see how classical fields are quantized to give their quantum counterparts. Our treatment proceeds in category theoretical setup and follows [6] and [2].

The category $\mathfrak{Man}$. This is the category of all n-dimensional C^∞ vector bundles $\mathcal{T} \xrightarrow{\pi} M$ with a pseudo-Riemannian fiber metric, a metric connection ∇, and a normal hyperbolic operator $D = \Box^\nabla + m^2$, over 4-dimensional oriented and time-oriented Lorentzian globally hyperbolic manifolds (e.g., our familiar bundle $\mathcal{T}$). They comprise the collection $\mathrm{Obj}(\mathfrak{Man})$. Then $\mathrm{Hom}(\mathfrak{Man})$ consists of C^∞ embeddings $\Psi : \mathcal{T}_1 \to \mathcal{T}_2$ which are vector bundle isometries (as defined earlier) preserving D and covering orientation and time-orientation preserving embeddings $\psi = \pi_2 \circ \Psi \circ \pi_1^{-1} : M_1 \to M_2$ with the property that

$\forall p, q \in \psi(M_1) \subset M_2$ with $q \in J^+(p)$ it follows $J^+(p) \cap J^-(q) \subset \psi(M_1)$. The essence of each requirement can be found in the above mentioned references. Note that because of the injectivity every morphism Ψ is monic. If it turns out to be epic as well, then it is surjective, and therefore an isomorphism. Thus $\mathfrak{Man}$ is a balanced category. $\mathfrak{Man}$ apparently does not have initial or terminal objects. Indeed, if a bundle $\mathcal{T}$ where a terminal object, then there should exist an isometric embedding into it of the same bundle with, say, completely different metric, which is in general not possible. On the other hand, if the bundle $\mathcal{T}$ was an initial object, then for any pair of morphisms $\Psi_{1,2} : \mathcal{T}_{1,2} \to \mathcal{T}_3$, with $\Psi_1(\mathcal{T}_1) \cap \Psi_2(\mathcal{T}_2) = \emptyset$ (which can always be found), there would exist two unique morphisms $\Phi_{1,2} : \mathcal{T} \to \mathcal{T}_{1,2}$, and therefore also two distinct morphisms $\Psi_{1,2} \circ \Phi_{1,2} : \mathcal{T} \to \mathcal{T}_3$, which is a contradiction.

The categories $\mathfrak{Alg}$ and $\mathfrak{TAlg}$. The collection $\mathrm{Obj}(\mathfrak{Alg})$ consists of all unital C^* algebras, and $\mathrm{Hom}(\mathfrak{Alg})$ contains all unit preserving injective $*$-homomorphisms. Similarly, $\mathrm{Obj}(\mathfrak{TAlg})$ is the collection of all topological unital $*$-algebras, and the members of $\mathrm{Hom}(\mathfrak{TAlg})$ are all homeomorphic unit preserving injective $*$-homomorphisms. Again, all morphisms in both categories are monic. Because the extension of a homomorphism from a proper subalgebra to the entire algebra is never unique, any epic morphism in either category is necessarily surjective. For C^* algebras, any unit preserving injective $*$-homomorphism is an isometric, in particular, homeomorphic (see, e.g., Proposition 4.1.22 [2]). Thus each epic (hence bijective) morphism in $\mathrm{Hom}(\mathfrak{Alg})$ is an isomorphism, so that $\mathfrak{Alg}$ is balanced. This is also true for general topological $*$-algebras when considering homeomorphic embeddings. $\mathfrak{Alg}$ has an initial object, which is the algebra of scalar operators $\mathbb{C}\mathbf{1}$. This is also true for $\mathfrak{TAlg}$, as the multiplication with a scalar is a continuous operation in any topological algebra.

Relations $\perp$ and $\overset{\mathrm{Dyn}}{\approx}$. For a category $\mathfrak{C}$ denote the category of right wedges (i.e., diagrams of the template $: \rightrightarrows \bullet$) of $\mathfrak{C}$ by $\mathfrak{C}^{\downarrow\downarrow}$. Any functor $F : \mathfrak{C}_1 \to \mathfrak{C}_2$ between to categories naturally induces a functor $\mathrm{Ind}^{\downarrow\downarrow} F : {\mathfrak{C}_1}^{\downarrow\downarrow} \to {\mathfrak{C}_2}^{\downarrow\downarrow}$. Let $\mathfrak{Man}_{\perp}^{\downarrow\downarrow}$ be the category $\mathfrak{Man}^{\downarrow\downarrow}$ equipped with the unary relation $\perp$ as follows. An element $(\mathcal{T}_1 \to \mathcal{T} \leftarrow \mathcal{T}_2) = a \in \mathrm{Obj}(\mathfrak{Man}_{\perp}^{\downarrow\downarrow})$ satisfies $a\perp$ if the images of $\mathcal{T}_1$ and $\mathcal{T}_2$ are causally disjoint in $\mathcal{T}$. Similarly let $\mathfrak{Alg}_{\perp}^{\downarrow\downarrow}$ and $\mathfrak{TAlg}_{\perp}^{\downarrow\downarrow}$ be the equipped categories $\mathfrak{Alg}^{\downarrow\downarrow}$ and $\mathfrak{TAlg}^{\downarrow\downarrow}$, respectively, where $a = (\mathcal{A}_1 \to \mathcal{A} \leftarrow \mathcal{A}_2)$ satisfies $a\perp$ if the images of $\mathcal{A}_1$ and $\mathcal{A}_2$ in $\mathcal{A}$ commute. Denote by $\mathfrak{Man}^{\overset{\mathrm{Dyn}}{\approx}}$ the subcategory of $\mathfrak{Man}$ where only those morphisms $\Psi : \mathcal{T}_1 \to \mathcal{T}_2$ are allowed for which $\pi_2 \circ \Psi \pi_1^{-1}(M_1)$ contains a Cauchy surface of M_2. Correspondingly, denote by $\mathfrak{Alg}^{\overset{\mathrm{Dyn}}{\approx}}$ ($\mathfrak{TAlg}^{\overset{\mathrm{Dyn}}{\approx}}$) the subcategory of $\mathfrak{Alg}$ ($\mathfrak{TAlg}$) which is totally disconnected, i.e., where only automorphisms are allowed.

Locally covariant quantum field theories. A locally generally covariant C^* ($*$, respectively) quantum field theory is a covariant functor LCQFT : $\mathfrak{Man} \to \mathfrak{Alg}$ (LCQFT : $\mathfrak{Man} \to \mathfrak{TAlg}$, respectively). Denote by $\mathfrak{C}^{op}$ the dual category of a category $\mathfrak{C}$. Equivalent formulations of a locally covariant quantum field theory would be a $\mathfrak{Alg}$ ($\mathfrak{TAlg}$) valued presheaf on $\mathfrak{Man}^{op}$ or precosheaf on $\mathfrak{Man}$, either a $\mathfrak{Alg}^{op}$ ($\mathfrak{TAlg}^{op}$) valued presheaf on $\mathfrak{Man}$. To check whether the latter presheaf is actually a sheaf, we need to verify the local identity and gluing axioms. In fact this is never the case, because the algebraic union of two distinct algebras is always greater then their set theoretical union. LCQFT is called causal if the induced functor $\mathrm{Ind}^{\sqcup}$LCQFT : $\mathfrak{Man}^{\sqcup}_{\perp} \to \mathfrak{Alg}^{\sqcup}_{\perp}$ ($\mathfrak{TAlg}^{\sqcup}_{\perp}$, respectively) is $\perp$-preserving. Further, LCQFT is said to obey the time-slice axiom if its restriction to the subcategory $\mathfrak{Man}^{\overset{\mathrm{Dyn}}{\approx}}$ has its image in $\mathfrak{Alg}^{\overset{\mathrm{Dyn}}{\approx}}$ ($\mathfrak{TAlg}^{\overset{\mathrm{Dyn}}{\approx}}$).

Locally covariant quantum fields. Let $\mathfrak{Test}$ be the category of test function spaces $\mathcal{D}(\mathcal{T}) \in \mathrm{Obj}(\mathfrak{Test})$ of all $\mathcal{T} \in \mathrm{Obj}(\mathfrak{Man})$, with morphisms being the pullbacks $\psi^* \in \mathrm{Hom}(\mathfrak{Test})$ of morphisms $\psi \in \mathrm{Hom}(\mathfrak{Man})$. Then the prescription $\mathcal{T} \to \mathcal{D}(\mathcal{T})$ defines a covariant functor TEST : $\mathfrak{Man} \to \mathfrak{Test}$. Our definition of a locally covariant quantum field will slightly differ from that of [6] in that we will also allow non observable fields [58],[2]. An example of a non observable field is the Dirac field. We will assume that there is a functorial way of relating the field algebra with the algebra of observables. This is not proven to be true for all quantum fields, but holds at least for the Dirac field. More precisely, we assume there exists a functor OBS : $\mathfrak{Alg} \to \mathfrak{Alg}$ (OBS : $\mathfrak{TAlg} \to \mathfrak{TAlg}$) which maps an observable algebra to the corresponding field algebra. The setup of [6] corresponds to the choice where OBS $= \mathbf{1}$, and this is true for the observable fields. Consider $\mathfrak{Test}$ and $\mathfrak{Alg}$ ($\mathfrak{TAlg}$) as subcategories of $\mathfrak{Top}$. Given a theory LCQFT, a locally covariant quantum field of LCQFT is a natural transformation Φ : TEST $\to$ LCQFT $\circ$ OBS, where the two functors are considered as TEST, LCQFT $\circ$ OBS : $\mathfrak{Man} \to \mathfrak{Top}$. This means, to each $\mathcal{T} \in \mathrm{Obj}(\mathfrak{Man})$ there exists a morphism from $\mathrm{Hom}(\mathfrak{Top})$ (i.e., a continuous map) $\Phi_{\mathcal{T}} : \mathcal{D}(\mathcal{T}) \to \mathfrak{Alg}$ (or $\mathfrak{TAlg}$). The field Φ is called linear if all these maps $\Phi_{\mathcal{T}}$ are algebra valued linear distributions. Define the category $\mathfrak{Test}^{\sqcup}_{\perp}$ by setting $\perp$ in $\mathfrak{Test}^{\sqcup}$ via the pullback of the map $\mathrm{Ind}^{\sqcup}$TEST. If LCQFT is causal and OBS $= \mathbf{1}$, then the functor $\mathrm{Ind}^{\sqcup}\Phi : \mathfrak{Test}^{\sqcup}_{\perp} \to \mathfrak{Alg}^{\sqcup}_{\perp}$ ($\mathfrak{TAlg}^{\sqcup}_{\perp}$) is again $\perp$ preserving. Such fields are also called causal. Similar statements can be made for the time-slice axiom.

The state space. Given an algebra $\mathcal{A} \in \mathrm{Obj}(\mathfrak{Alg})$ (or $\mathcal{A} \in \mathrm{Obj}(\mathfrak{TAlg})$) a state ω on $\mathcal{A}$

is a continuous linear functional $\omega \in \mathcal{A}'$ (here by $\mathcal{A}'$ we do not mean the commutant of $\mathcal{A}$ as it is usually done in the algebraic literature), which is positive, $\omega(A^*A) \geq 0$, and normalized, $\omega(\mathbf{1}) = 1$. The space of all states on $\mathcal{A}$ is a convex linear space. This space is usually too large in the sense that it contains states which have no reasonable physical interpretation. Therefore one chooses a subspace $\mathrm{Sts}(\mathcal{A})$ such that its members satisfy some reasonable conditions, e.g., local quasi-equivalence, normality and intermediate factoriality (see [6] for details). A state $\omega \in \mathrm{Sts}(\mathcal{A})$ is called pure if it is extremal, i.e., lies on the boundary of the convex space. If $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ is a morphism, then it needs not induce a map $\phi^* : \mathrm{Sts}(\mathcal{A}_1) \to \mathrm{Sts}(\mathcal{A}_2)$ because it may not respect the imposed properties of a state. Therefore we start with a subcategory $\mathfrak{Alg}_+^{op} \subset \mathfrak{Alg}^{op}$ ($\mathfrak{TAlg}_+^{op} \subset \mathfrak{TAlg}^{op}$) which is compatible with the desired properties of states. Now define the category $\mathfrak{Sts}$ ($\mathfrak{TSts}$), isomorphic to $\mathfrak{Alg}_+^{op}$ ($\mathfrak{TAlg}_+^{op}$), as follows. To each $\mathcal{A} \in \mathrm{Obj}(\mathfrak{Alg}_+^{op})$ ($\mathcal{A} \in \mathrm{Obj}(\mathfrak{TAlg}_+^{op})$) put in correspondence $\mathrm{DUAL}(\mathcal{A}) = \mathrm{Sts}(\mathcal{A}) \in \mathrm{Obj}(\mathfrak{Sts})$ ($\mathrm{DUAL}(\mathcal{A}) = \mathrm{Sts}(\mathcal{A}) \in \mathrm{Obj}(\mathfrak{TSts})$), and to each morphism $\phi^{-1} \in \mathrm{Hom}(\mathfrak{Alg}_+^{op})$ ($\phi^{-1} \in \mathrm{Hom}(\mathfrak{TAlg}_+^{op})$) assign the pullback map $\mathrm{DUAL}(\phi^{-1}) = \phi^* \in \mathrm{Hom}(\mathfrak{Sts})$ ($\mathrm{DUAL}(\phi^{-1}) = \phi^* \in \mathrm{Hom}(\mathfrak{Sts})$). If the properties of the states we imposed are generally covariant, then given a locally covariant theory $\mathrm{LCQFT} : \mathfrak{Man} \to \mathfrak{Alg}$ ($\mathfrak{TAlg}$), the contravariant counterpart of its dual functor, $\mathrm{LCQFT}_c^{op} : \mathfrak{Man} \to \mathfrak{Alg}^{op}$ ($\mathfrak{TAlg}^{op}$), in fact takes values inside $\mathfrak{Alg}_+^{op}$ ($\mathfrak{TAlg}_+^{op}$). Then LCQFT_c^{op} composed with the isomorphism DUAL described above, gives a contravariant functor $\mathrm{STAT} : \mathfrak{Man} \to \mathfrak{Sts}$ ($\mathrm{STAT} : \mathfrak{Man} \to \mathfrak{TSts}$), i.e., $\mathrm{STAT} = \mathrm{DUAL} \circ \mathrm{LCQFT}_c^{op}$. This functor STAT is called the state space for the theory LCQFT.

The CCR quantization. As a first step in constructing concrete quantum fields it is natural to look for quantum counterparts of classical linear hyperbolic fields. This procedure is called quantization. Quantum statistics suggests that any quantum field should satisfy either Bose-Einstein or Fermi-Dirac statistics. Then spin-statistics theorems show that quantized integer spin hyperbolic fields can obey only Bose-Einstein statistics, and half-integer spin (spinor) fields only Fermi-Dirac statistics (see [58] for a rigorous statement on the Klein-Gordon field). Two distinguished realizations of the Bose-Einstein and Fermi-Dirac statistics are the canonical commutation relations (CCR) and canonical anti-commutation relations (CAR), respectively. As we did not have the opportunity to introduce spinor field in our classical part (they can be considered as linear hyperbolic fields with additional spinor structure), in what follows we will concentrate on the CCR quantization and CCR quantum field theories with the hope to return to the CAR theories in the future.

One of the ways of constructing quantum fields is the so called Borchers-Uhlmann algebra [4]. Let a linear hyperbolic field give rise to a category $\mathfrak{Man}$. We construct the corresponding locally covariant quantum field theory LCQFT : $\mathfrak{Man} \to \mathfrak{TAlg}$ as follows. For each $\mathcal{T} \in \mathrm{Obj}(\mathfrak{Man})$ we set LCQFT($\mathcal{T}$) to be the Borchers-Uhlmann algebra $\mathcal{B}(\mathcal{T})$ constructed upon $\mathcal{D}(\mathcal{T})/\ker E$ similar to [50] divided by the CCR relation $[A(f), A(g)] + i\langle E[f], g\rangle_M = 0$, where the generators are given by $A(f) = (0, f, 0, ...)$ for all $f \in \mathcal{D}(\mathcal{T})/\ker E$. $\mathcal{B}(\mathcal{T})$ is then a topological $*$-algebra, which is isotonous, i.e., for any morphism $\psi : \mathcal{T}_1 \to \mathcal{T}_2$ there is an embedding $\phi : \mathcal{B}(\mathcal{T}_1) \to \mathcal{B}(\mathcal{T}_2)$. This allows us to set LCQFT(ψ) $= \psi^*$, the pullback map. (The covariance of the propagator E is used here tacitly, which follows from the covariance of D.) This quantum field theory is automatically causal by the properties of E. Now the corresponding locally covariant quantum field Φ is constructed by defining $\Phi_{\mathcal{T}} : \mathcal{D}(\mathcal{T}) \to \mathcal{B}(\mathcal{T})$ by $\Phi_{\mathcal{T}}(f) = A(f + \ker E)$. That this is indeed a natural transformation can be seen easily.

Finally we come to the state space. We readily obtain a state space for LCQFT when we fix the isomorphism DUAL. Let us recall a few definitions. Let $\Phi_{\mathcal{T}} : \mathcal{D}(\mathcal{T}) \to \mathcal{A}$ describe a quantum field, and let ω be a state on $\mathcal{A}$. We define the n-point functions of ω by

$$\omega_n(f_1, ..., f_n) = \omega(\Phi_{\mathcal{T}}(f_1)...\Phi_{\mathcal{T}}(f_n)),\ \forall n \in \mathbb{N}_0,\ f_1, ..., f_n \in \mathcal{D}.$$

Due to the topology of Borchers-Uhlmann algebra, all n-point functions are n-distributions $\omega_n \in (\bigotimes_n \mathcal{D}(\mathcal{T}))$. The state ω will be called quasifree if

$$\omega_{2n+1}(f_1, ...f_{2n+1}) = 0,\ \omega_{2n+2}(f_1, ..., f_{2n+2}) = \sum_{s\in[s]} \prod_{j=1}^{n+1} \omega_2(f_{s(j)}, f_{s(n+1+j)}),\ \forall n \in \mathbb{N}_0,$$

where $[s]$ is the set of all permutations satisfying $s(1) < s(2) < ... < s(n+1)$ and $s(j) < s(n+1+j)$. A quasifree state ω is said to be Hadamard, if the wave front set of its 2-point function satisfies the microlocal spectral condition (see [51]). We will have the occasion to consider these states in more detail. Next, to each positive functional ω on a $*$-algebra $\mathcal{A}$ there exists a distinguished (up to unitary equivalence) cyclic $*$-representation π_ω called the GNS (or Wightman) representation with the representation Hilbert space $(\mathcal{H}_\omega, (,)_\omega)$ and a unit cyclic vector Ω_ω such that $\omega(A) = (\Omega_\omega, \pi_\omega(A)\Omega_\omega)_\omega$ for all $A \in \mathcal{A}$ of the form $A = e^{i\Phi_{\mathcal{T}}(f)}$ with $f \in \mathcal{D}$ (see [46],[1]). The folium of the representation π_ω is the set of all states ω' on $\mathcal{A}$ for which there exists a trace class operator $\rho_{\omega'} : \mathcal{H}_\omega \to \mathcal{H}_\omega$ (density matrix) with $\omega'(A) = Tr[\rho_{\omega'}\pi_\omega(A)]$. Two states are called quasi-equivalent, if the folia of their GNS representations coincide. It is generally believed that all quasifree Hadamard states are locally quasi-equivalent for all vector bundle fields. With all this in mind we define the isomorphism DUAL by setting DUAL($\mathcal{A}$) to be the folium of a quasifree Hadamard state ω on $\mathcal{A}$. That this indeed gives an isomorphism of categories, and that the state space thus defined satisfies the above mentioned desired properties, has been proven in [57] for the Klein-Gordon field. A folk wisdom says that this should be the case in general as well. Therefore henceforth we will mainly concentrate on quasi-free Hadamard states.

4.2 The structure of 2-point functions

In this section we will obtain a general form for the 2-point function of a state ω on the field algebra $\mathcal{A}$ using the mode decomposition. We suppose that the assumptions of **Proposition 1.9** are satisfied. Assume further, that the constraints on the choice of the modes T_α and on the spatial Fourier transform are the same as in the section about the propagator.

The matrix notation. First we establish some matrix notations, which will be reminiscent of those of [32], and they are indeed related. We could use the explicit formula for the propagator to transfer from a covariant picture to the canonical picture on a Cauchy surface, and that is done in [32]. A transition from the canonical matrices of [32] to covariant matrices has been done in [52] for the homogeneous isotropic quasifree states of the scalar field on FRW spacetimes. Our matrices will slightly differ from those of Schlemmer and will be applied in the full generality.

The mode decomposition $f \to \tilde{f} = \tilde{f}^u \oplus \tilde{f}^v = u_\alpha(f) \oplus v_\alpha(f)$ gives a surjective linear map $\mathcal{D}(\mathcal{T}) \to \tilde{\mathcal{K}} = \tilde{\mathcal{D}}^u(\tilde{\Sigma}) \oplus \tilde{\mathcal{D}}^v(\tilde{\Sigma})$ which has the kernel $\ker E$. Denote $\mathcal{K} = \widetilde{\mathcal{D}(\mathcal{T})/}\ker E$. Any linear map $\mathbf{S} : \mathcal{K} \to \mathcal{D}(\mathcal{T})$ has its pullback image $\tilde{\mathbf{S}} : \tilde{\mathcal{K}} \to \tilde{\mathcal{K}}$ with $[\mathbf{S}(f + \ker E)] = \tilde{\mathbf{S}}\tilde{f}$, and all linear maps $\tilde{\mathbf{S}} : \tilde{\mathcal{K}} \to \tilde{\mathcal{K}}$ arise in this way. A weak solution $\phi \in \mathcal{D}(\mathcal{T})'$ of the field equation is precisely a distribution $\phi \in \mathcal{K}'$, and the essence of the mode decomposition is that there exists a bijective linear map $\phi \to \tilde{\phi} = a^\phi \oplus b^\phi \in \tilde{\mathcal{K}}' = \tilde{\mathcal{D}}^u(\tilde{\Sigma})' \oplus \tilde{\mathcal{D}}^v(\tilde{\Sigma})'$ such that $\phi(f) = \tilde{\phi}(\tilde{f}) = a^\phi(\tilde{f}^u) + b^\phi(\tilde{f}^v)$. Introduce the following pairing on $L^2(\tilde{\Sigma}, d\mu)$,

$$\langle \tilde{f}, \tilde{g} \rangle_s = \int_{\tilde{\Sigma}} d\mu(\alpha) s(\alpha) \tilde{f}(-\alpha) \tilde{g}(\alpha), \tag{4.1}$$

which is in fact the Fourier counterpart of $\langle , \rangle_\Sigma$. As $\tilde{\mathcal{D}}^u(\tilde{\Sigma}), \tilde{\mathcal{D}}^v(\tilde{\Sigma}) \subset L^2(\tilde{\Sigma}, d\mu)$, we can interpret any $\tilde{\phi} \in L^1_{loc}(\tilde{\Sigma}, d\mu)$ as a distribution in $\tilde{\mathcal{D}}^u(\tilde{\Sigma})'$ or $\tilde{\mathcal{D}}^v(\tilde{\Sigma})'$ by setting $\tilde{\phi}(\tilde{f}^u) =$

$\langle f^u, \phi\rangle_s$ and $\tilde{\phi}(\tilde{f}^v) = \langle f^v, \phi\rangle_s$, correspondingly. Thus we have $L^1_{loc}(\tilde{\Sigma}, d\mu) \oplus L^1_{loc}(\tilde{\Sigma}, d\mu) \subset \tilde{\mathcal{K}}'$. Denote $\tilde{f}_-(\alpha) = \tilde{f}(-\alpha)$ whenever applicable.

Now to each weak solution ϕ and to its mode decomposition $\tilde{\phi} \in \tilde{\mathcal{K}}'$ we assign a column $\check{\phi} = (a^\phi, b^\phi)^\top$, whereas to each mode decomposed test function $\tilde{f} \in \tilde{\mathcal{K}}$ we assign a row $\check{f}^\top = (\tilde{f}^u, \tilde{f}^v)$. The multiplication of a row with a column is defined naturally, $\check{f}^\top \check{\phi} = \tilde{\phi}(\tilde{f})$. The substitution $\alpha \to -\alpha$ in (Eq.4.1) shows that we could equally multiply a distribution from the left with a test function from the right, but it is important that one side always represents a distribution and the other side a test function. Because $\overline{\tilde{\bar{f}}^u}(\alpha) = \tilde{f}^v(-\alpha)$, we have $\overline{\check{\bar{f}}^\top} = \check{f}_-^\top \hat{\tau}$, where

$$\hat{\tau} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

A linear map $\tilde{\mathbf{S}} : \tilde{\mathcal{K}} \to \tilde{\mathcal{K}}'$ is given by a 2×2 matrix $\hat{\mathbf{S}}$ as

$$\hat{\mathbf{S}}\check{f} = \begin{pmatrix} \tilde{\mathbf{S}}^{u,u} & \tilde{\mathbf{S}}^{u,v} \\ \tilde{\mathbf{S}}^{v,u} & \tilde{\mathbf{S}}^{v,v} \end{pmatrix} \begin{pmatrix} \tilde{f}^u \\ \tilde{f}^v \end{pmatrix},$$

where each entry is a linear map, and the cross terms appear because $\tilde{\mathcal{D}}^u(\tilde{\Sigma}) \cap \tilde{\mathcal{D}}^v(\tilde{\Sigma}) \neq \emptyset$. Any weak bi-solution $\omega_2 \in (\mathcal{D}(\mathcal{T}) \otimes \mathcal{D}(\mathcal{T}))'$ can be written as a continuous map $\omega_2 : \mathcal{K} \to \mathcal{K}'$ such that $\omega_2(f, g) = \omega_2[g](f)$. Therefore for each g it has a column $(\widecheck{\omega_2[g]})$ such that $\omega_2(f, g) = \check{f}^\top(\widecheck{\omega_2[g]})$. On the other hand, the mode decomposed image of the linear map ω_2 gives a matrix $\hat{\omega}$ such that $(\widecheck{\omega_2[g]}) = \hat{\omega}\check{g}$. Finally we get $\omega_2(f, g) = \check{f}^\top \hat{\omega} \check{g}$. Note that both multiplications are distributional actions, and each element of the matrix $\hat{\omega}$ is a bi-distribution. However, only one of two distributional actions we will consider as corresponding to the pairing $\langle , \rangle_s$. The other one will be understood as by usual real-linear L^2-product. That means, if we instead of a bi-distribution write its kernel, then we mean that it acts by one pairing on the left and by the other on the right. Such a bizarre notation is chosen to make the formulae readable and precise at the same time. We hope that the intuition that a 2-point function is merely a bi-linear map of whatever form will prevent the reader from being confused with this notation. One can always translate everything into the language of four distributional coefficients of the mode decomposed bi-solution. If the map $\mathbf{S}$ is such that $\omega_2(f, \mathbf{S}g)$ (or $\omega_2(\mathbf{S}f, g)$) makes sense as a distribution, then we will write $\hat{\omega}\hat{\mathbf{S}}$ (respectively $\hat{\mathbf{S}}^\top \hat{\omega}$) to denote its matrix.

In particular, we denote by $\hat{\sigma}$ the symplectic matrix of the bi-solution $-i\langle E[f], g\rangle_M$,

$$\hat{\sigma} = \begin{pmatrix} 0 & -\delta \\ \delta & 0 \end{pmatrix},$$

where $\delta(\alpha - \beta)$ is the kernel of the well known bi-distribution.

The 2-point functions of quasifree states. The 2-point function ω_2 of a quasifree state ω is a weak bi-solution of the field equation which in addition is hermitian, positive and satisfies CCR. And conversely, any such bi-solution can be used to compute all even n-point functions, and thus to determine a quasifree state.

Proposition 4.1 *The 2-point function ω_2 of quasifree state is given by a matrix*

$$\hat{\omega} = \begin{pmatrix} a^\omega & b^\omega \\ \bar{b}^\omega_{--} + \delta & \bar{a}^\omega_{--} \end{pmatrix},$$

where coefficients satisfy $a^\omega - a^{\omega\top} = 0$, $b^\omega = b^\omega_{--}{}^*$, $b^\omega_{-+}(\overline{\tilde{f}^u}, \tilde{f}^v) \geq 0$ and

$$b^\omega_{+-}(\tilde{f}^u, \overline{\tilde{f}^v}) + \delta_{-+}(\overline{\tilde{f}^v}, \tilde{f}^u) \geq 0,$$

$$|a^\omega_{-+}(\overline{\tilde{f}^u}, \tilde{f}^u)|^2 \leq b^\omega_{-+}(\overline{\tilde{f}^u}, \tilde{f}^v) \left(b^\omega_{+-}(\tilde{f}^u, \overline{\tilde{f}^v}) + \delta_{-+}(\overline{\tilde{f}^v}, \tilde{f}^u) \right).$$

Proof: Let

$$\hat{\omega} = \begin{pmatrix} a^\omega & b^\omega \\ c^\omega & d^\omega \end{pmatrix}$$

be the matrix of the 2-point function ω_2. The hermiticity of ω_2 means $\omega_2(f, g) = \overline{\omega_2(\bar{g}, \bar{f})}$. We have $\check{\bar{g}}^\top = \overline{\check{g}^\top_-}\hat{\tau}$ and $\check{\bar{f}} = \hat{\tau}\overline{\check{f}_-}$, thus

$$\omega_2(\bar{g}, \bar{f}) = \overline{\check{g}^\top_-}\hat{\tau}\hat{\omega}\hat{\tau}\overline{\check{f}_-} = \overline{\check{f}^\top}\hat{\tau}\hat{\omega}^\top_{--}\hat{\tau}\overline{\check{g}},$$

and we find $\hat{\omega} = \hat{\omega}^*_{--}$. In components this means $a^\omega = d^\omega_{--}{}^*$, $b^\omega = b^\omega_{--}{}^*$ and $c^\omega = c^\omega_{--}{}^*$. The positivity of ω_2 tells that $\omega_2(\bar{f}, f) \geq 0$. Thus we establish that

$$\overline{\check{f}^\top_-}\hat{\tau}\hat{\omega}\check{f} = \overline{\check{f}^\top}\hat{\tau}\hat{\omega}_{-+}\check{f} \geq 0,$$

which as usual means $c^\omega_{-+}, b^\omega_{-+} \geq 0$ and $\det \hat{\omega}_{-+} = c^\omega_{-+}b^\omega_{-+} - a^\omega_{-+}d^\omega_{-+} \geq 0$, where everywhere evaluation on $(\bar{\tilde{f}}, \tilde{f})$ is understood. The satisfaction of CCR has the form

$\omega_2(f,g) - \omega_2(g,f) = -i\langle E[f], g\rangle_M$, which in the matrix language becomes $\hat{\omega} - \hat{\omega}^\top = \sigma$, or in components $a^\omega - a^{\omega\top} = d^\omega - d^{\omega\top} = 0$ and $c^\omega - b^{\omega\top} = \delta$. The combination of all these results entails the assertion. □

A quasifree state gives an inner product $(f,g)^s_\omega = \omega(\bar{f}, g) + \omega(g, \bar{f})$ on $\mathcal{D}(\mathcal{T})$, and we will denote the Hilbert space completion by $\mathfrak{h}^s_\omega$. The following proposition is a simple adaptation of some terminology of [1].

Proposition 4.2 *A quasifree state ω is pure if and only if there exists a bounded operator $\mathbf{S} \in \mathcal{B}(\mathfrak{h}^s_\omega)$ such that either of the following holds.*

(i) $\omega_2(f, \mathbf{S}g) = \omega_2(f,g)$ and $\omega_2(\mathbf{S}f, g) = 0$, $\forall f, g \in \mathcal{D}(\mathcal{T})$

(ii) $\omega_2(f,g) = -i\langle E[f], \mathbf{S}g\rangle_M$ and $\mathbf{S}^2 = \mathbf{S}$, $\forall f, g \in \mathcal{D}(\mathcal{T})$

Proof: For the characterization of pure states we use the technique of [1]. The quotient test function space $\mathcal{K} = \mathcal{D}(\mathcal{T})/\ker E$ equipped with the complex conjugation $\Gamma f = \bar{f}$ and the hermitian form $\gamma(f,g) = -i\langle E[\bar{f}], g\rangle_M$ represents a phase space $(\mathcal{K}, \Gamma, \gamma)$ in the sense of [1]. If ω_2 is the 2-point function of the state ω, then $S(f,g) = \omega_2(\bar{f}, g)$ is a polarization in $(\mathcal{K}, \Gamma, \gamma)$. Then $\mathfrak{h}^s_\omega = \overline{\mathcal{K}}^{(,)S}$ by formula 3.4 of [1]. By definition 3.11 in [1] the polarization S is called generalized Fock polarization if the spectrum of the operator $\gamma_{\mathbf{S}}$ defined by formula 3.7 of the same paper is contained in $\{-1, 0, 1\}$. In the context of algebraic quantum field theory, a quasifree state ω is pure if and only if it is Fock. Thus we have that $\mathrm{Spec}(\gamma_{\mathbf{S}}) \subset \{-1, 0, 1\}$. But from formula 3.7 in [1] it is clear that because $\langle E[\hat{f}], g\rangle_M$ is non-degenerate on $\mathcal{K}$, $\gamma_{\mathbf{S}}$ cannot have a null space, $\mathrm{Spec}(\gamma_{\mathbf{S}}) \subset \{-1, 1\}$. Then by formula 3.8 in this paper we find $\mathbf{S} = 1/2(1 + \gamma_{\mathbf{S}})$, and thus $\mathrm{Spec}(\mathbf{S}) = \{0, 1\}$. Thereby we have established that ω is pure if and only if $\mathbf{S}$ is a projection (this conclusion is presented as being obvious in [32]).

If ω is pure, then $\mathbf{S}$ is a projection and hence $(f, \mathbf{S}g)_S = (f, \mathbf{S}^2 g)_S$, which by $S(f,g) = (f, \mathbf{S}g)_S$ is equivalent to $S(f,g) = S(f, \mathbf{S}g)$. It also follows $S(\mathbf{S}g, f) = 0$. But CCR require $S(f, \mathbf{S}g) - S(\mathbf{S}g, f) = \gamma(f, \mathbf{S}g)$. Thus both (i) and (ii) follow from the fact that ω is pure. Now suppose (ii) holds. Then $S(f,g) = \gamma(f, \mathbf{S}g)$, and $S(f, \mathbf{S}g) = \gamma(f, \mathbf{S}^2 g) = \gamma(f, \mathbf{S}g) = S(f,g)$, and from CCR we find $S(\mathbf{S}g, f) = 0$. This means (i) follows from (ii). If we suppose that (i) holds, then from $(f,g)_S = S(f,g) + S(g,f)$ it follows that $(f, \mathbf{S}g)_S = S(f, \mathbf{S}g) + S(\mathbf{S}g, f) = S(f, \mathbf{S}g) = S(f,g)$, thus $\mathbf{S}$ is indeed the operator defined in formula 3.7. And because $(f, \mathbf{S}^2 g)_S = S(f, \mathbf{S}g) = S(f,g) = (f, \mathbf{S}g)_S$ we find that $\mathbf{S}$ is a projection, i.e., ω is pure. □

Corollary 4.1 *A quasifree pure state ω is given by a matrix $\hat{\omega} = \hat{\sigma}\hat{\mathbf{S}}$, where*

$$\hat{\mathbf{S}} = \begin{pmatrix} 1 - \overline{\tilde{S}^{v,v}} & -\overline{\tilde{S}^{v,u}} \\ \tilde{S}^{v,u} & \tilde{S}^{v,v} \end{pmatrix}.$$

The linear maps $\tilde{S}^{v,u}$, $\tilde{S}^{v,v}$ satisfy

$$\delta\tilde{S}^{v,v} = \tilde{S}^{v,v*}\delta,\ \delta\tilde{S}^{v,u} = \tilde{S}^{v,u\top}\delta,\ \tilde{S}^{v,v}(1-\tilde{S}^{v,v}) = \tilde{S}^{v,u}\overline{\tilde{S}^{v,u}},\ \tilde{S}^{v,v}\tilde{S}^{v,u} = \tilde{S}^{v,u}\overline{\tilde{S}^{v,v}},$$

$$-\delta_{-+}\tilde{S}^{v,v} \geq 0,\ \delta_{-+}(1-\overline{\tilde{S}^{v,v}}) \geq 0,\ -\delta_{-+}\tilde{S}^{v,v}\delta_{-+}(1-\overline{\tilde{S}^{v,v}}) + \delta_{-+}\tilde{S}^{v,u}\delta_{-+}\overline{\tilde{S}^{v,u}} \geq 0,$$

where again the action of δ is understood by different pairings on the left and on the right.

Proof: The statement (ii) of the last proposition is written in the matrix language as $\hat{\omega} = \sigma\hat{\mathbf{S}}$. Now the positivity, hermiticity and CCR properties of $\hat{\omega}$, along with the condition that $\mathbf{S}$ is a projection, yield the formulas. □

Note that there is a large arbitrariness in the choice of the modes T_α. One can always switch to another family S_α, which is related with the old one by $T_\alpha = \mu_\alpha S_\alpha + \nu_\alpha \bar{S}_\alpha$. The restriction that the $S_\alpha = S_{-\alpha}$ gives $\mu_\alpha = \mu_{-\alpha}$ and $\nu_\alpha = \nu_{-\alpha}$. For the Wronskian to equal i it needs to hold $|\mu_\alpha|^2 - |\nu_\alpha|^2 = 1$. It remains to assure that (Eq.1.19) holds for S_α as well. Under the assumptions of **Proposition 1.6** it suffices that $\mu_\alpha = \mu(\omega(\alpha))$ and $\nu_\alpha = \nu(\omega(\alpha))$ for some $\mu, \nu \in \mathcal{A}[\mathbb{H}_0]$ (do not confuse with the spectral measure $d\mu(\alpha)$). The new distributions are related with old ones as $u_\alpha = \mu_\alpha u'_\alpha + \nu_\alpha v'_\alpha$ and $v_\alpha = \bar{\nu}_\alpha u'_\alpha + \bar{\mu}_\alpha v'_\alpha$. In matrix notations $\check{f} = \hat{\mu}\check{f}'$, where

$$\hat{\mu} = \begin{pmatrix} \mu_\alpha & \nu_\alpha \\ \bar{\nu}_\alpha & \bar{\mu}_\alpha \end{pmatrix}.$$

Hence $\hat{\omega}$ transforms as $\hat{\omega}' = \hat{\mu}^\top \hat{\omega}\hat{\mu}$. This transformations are tightly related to Bogoliubov transformations, and we hope to have the occasion to turn to this relation afterwards.

4.3 Invariant quasifree states in semidirect homogeneous spaces

Of particular importance are the states which carry at least the same symmetries as the underlying vector bundle $\mathcal{T}$. We will assume that $\mathcal{T}$ is a semidirect homogeneous vector bundle in the sense of the second chapter, and will use the results obtained therein. We further assume that the adapted Fourier transform is conventional. As we already know this is the situation in FRW and Bianchi I-VII models.

Recall that in a semidirect homogeneous space $\alpha = (\pi, \lambda, r, s)$, and let symbolically $-\alpha = (-\pi, \lambda, -r, -s)$. Recall also that $\bar{\tilde{f}}(\alpha) = \tilde{f}(-\alpha)$. The following proposition characterizes the mode coefficients $a^\omega, b^\omega, c^\omega, d^\omega$ appearing in the matrix

$$\hat{\omega} = \begin{pmatrix} a^\omega & b^\omega \\ c^\omega & d^\omega \end{pmatrix}$$

of a homogeneous distributional bi-solution ω_2 of the field equation.

Proposition 4.3 *A bi-solution ω_2 of the field equation admits all the isometries of the G/O-homogeneous bundle $\mathcal{T}$ if and only if its coefficient distributions $a^\omega_j = a^\omega, b^\omega, c^\omega, d^\omega$ $(j = 1, ..., 4)$ are given by distributions $\mathfrak{a}^\omega_j(\pi, \lambda, s, \lambda', s')$ so that*

$$a^\omega_j(\pi, \lambda, r, s, \pi', \lambda', r', s') = \delta(\pi + \pi', r + r')\mathfrak{a}^\omega_j(\pi, \lambda, s, \lambda', s')$$

in sense of kernels. If moreover all multiplicities $mult(\pi, U_g)$ are finite, then this distributions $\mathfrak{a}^\omega_j$ are given by μ-locally integrable fields of $[mult(\pi, U_g) \cdot n] \times [mult(\pi, U_g) \cdot n]$ complex matrices.

Proof: If ω_2 admits the isometries of $\mathcal{T}$ then it is (bi-)invariant under the quasi-regular action $U^{\mathcal{T}}_g$ of the spatial isometry group G,

$$\omega_2(U^{\mathcal{T}}_g f, U^{\mathcal{T}}_g h) = \omega_2(f, h), \forall f, h \in \mathcal{D}(\mathcal{T}), \forall g \in G.$$

Consider the restriction of ω_2 to $C^\infty_0(\mathcal{I}) \otimes \mathcal{D}(\mathcal{T}_t) \otimes C^\infty_0(\mathcal{I}) \otimes \mathcal{D}(\mathcal{T}_t)$, i.e, write $f(x) = f_0(t)f_1(\vec{x})$ and $h(x) = h_0(t)h_1(\vec{x})$. Because the isometries are purely spatial, we have $U^{\mathcal{T}}_g f_0(t)f_1(\vec{x}) = f_0(t)U^{\mathcal{T}}_g f_1(\vec{x})$ and $U^{\mathcal{T}}_g h_0(t)h_1(\vec{x}) = h_0(t)U^{\mathcal{T}}_g h_1(\vec{x})$. Fix f_0 and h_0, and consider $\omega_2(f,h) = \omega_2(f_0 f_1, h_0 h_1) = w[f_0,h_0](f_1,h_1)$ with some bi-distribution $w[f_0,h_0] \in (\mathcal{D}(\mathcal{T}_t) \otimes \mathcal{D}(\mathcal{T}_t))'$.Then we find

$$w[f_0,h_0](U_g^{\mathcal{T}} f_1, U_g^{\mathcal{T}} h_1) = w[f_0,h_0](f_1,h_1), \tag{4.2}$$

$w[f_0,h_0]$ is an invariant bi-distribution on $\mathcal{T}_t$ in sense of the chapter 2. On the other hand we see that

$$u_\alpha(f) = \int_{\mathcal{I}} dt T_\alpha(t) f_0(t) \tilde{f}_1(-\alpha) \doteq T_\alpha(f_0) \tilde{f}_1(-\alpha),$$
$$v_\alpha(f) = \int_{\mathcal{I}} dt \bar{T}_\alpha(t) f_0(t) \tilde{f}_1(-\alpha) \doteq \bar{T}_\alpha(f_0) \tilde{f}_1(-\alpha),$$

therefore $\check{f} = \tilde{f}_1(-\alpha) \check{T}(f_0)$, where we denote $\check{T}(f_0) = (T_\alpha(f_0), \bar{T}_\alpha(f_0))^\top$. Using **Proposition 4.1** we find that

$$w[f_0,h_0](f_1,h_1) = \tilde{w}[f_0,h_0](\tilde{f}_1 \otimes \tilde{h}_1),$$

where the bi-distribution $\tilde{w}[f_0,h_0]$ is given by the kernel

$$\hat{w}[f_0,h_0] = \check{T}(f_0)^\top \hat{\omega} \check{T}(h_0).$$

Now because (Eq.4.2) holds for arbitrary f_0, h_0, and that $T_\alpha(f_0)$, $\bar{T}_\alpha(f_0)$, $T_\alpha(h_0)$ and $\bar{T}_\alpha(h_0)$ are independent quantities, it follows that each component of $\hat{\omega}$ is individually invariant.

Now **Corollary 2.1** is applicable. Because D_π is an injective operator (see [21]) we can write $\pi(\bar{f})^* = D^{-1} \hat{f}(\pi)^*$ and for any invariant bi-distribution w find $\tilde{w}^{ij}$ such that

$$w(f,h) = \sum_{i,j} \tilde{w}^{ij}(\hat{f}_i(\pi)^* \hat{h}_i(\pi)) \doteq \tilde{w}(\hat{f}(\pi)^* \hat{h}(\pi)).$$

Using the fact that r parameterizes rows whereas λ and s parameterize columns, we find

$$\hat{f}(\pi)^* \hat{h}(\pi) = \int_{R_\pi} dr \tilde{f}(-\pi, \lambda, -r, -s) h(\pi, \lambda', r, s'),$$

and applying this to invariant bi-distributions $T_\alpha(f_0) T_\beta(h_0) a_\omega$ and $T_\alpha(f_0) \bar{T}_\beta(h_0) b_\omega$ we arrive at the desired result.

Now if $mult(\pi, U_g) < \infty$ then even **Proposition 2.6** can be applied, which immediately yields the assertion. The converse statement is obvious. $\square$

Remark 4.1 *If ω_2 is the 2-point function of a quasifree state, then, of course, the distributions a_s^ω, b_s^ω satisfy the corresponding symmetry, hermiticity and positivity conditions of* **Proposition 4.1**.

In FRW-homogeneous bundles G is unimodular and $mult(\pi, U_g) = 1$ for all π. Thus the fields A^ω, B^ω are μ-locally integrable $n \times n$ matrix functions of π. Since the Fourier image of the 2-point function is locally integrable, we can extend its domain to the test functions of which the Fourier transforms are locally integrable (and hence not necessarily compactly supported) functions of rapid decay in π. This allows also to weaken the conditions on the choice of the initial data for modes. Now they can be chosen locally integrable and polynomially bounded in π. Indeed, in that case (Eq.1.19) is satisfied for the locally integrable test function space. But then we can also perform transformations with μ_α, ν_α being locally integrable and polynomially bounded. In [32] and in the literature that refers to it one imposes an additional continuity requirement on ω_2 which implies that A^ω, B^ω are polynomially bounded in π. Then the magnitudes of the fields A^ω, B^ω are comparable with those of μ_α, ν_α. This allows, in particular, to pose the following problem. Given an invariant state ω find a transformation $\hat{\mu}$ such that in the new modes $A^\omega = 0$. This condition represents an $n \times n$ matrix equation involving unknown diagonal matrices $\text{diag}(\nu(\pi, i)/\mu(\pi, i))$. Because A^ω is a priori symmetric, the number of independent equations is $m(m+1)/2$, and the number of complex unknowns is m, where m is the minimal rank of the matrices A^ω, B^ω. Therefore a general solution exists if $m \leq 1$. As in general m can be equal n, this procedure is possible for $n = 1$. This has been done for the scalar field in [32] and [52]. However, if for some state such a procedure is successfully performed then the following remark gives a convenient characterization of purity.

Remark 4.2 *If ω is pure and $a_\omega = 0$ then by* **Corollary 4.1** *$\tilde{S}^{v,u} = 0$ and hence $\tilde{S}^{v,v}$ is a projection. But from $-\delta_{-+}\tilde{S}^{v,v} \geq 0$ it follows that $\overline{\tilde{S}^{v,v}} = 0$ whenever $s(\alpha) = 1$. On the other hand from $\delta_{-+}(1 - \overline{\tilde{S}^{v,v}}) \geq 0$ it follows that $\overline{\tilde{S}^{v,v}} - 1 \geq 0$ whenever $s(\alpha) = -1$. For a projection this means $\overline{\tilde{S}^{v,v}} - 1 = 0$. Thus $B^\omega = (1-s)/2$ and $C^\omega = (1+s)/2$.*

4.4 The Hadamard regularization of quasifree states

As already mentioned in the introductory section, the dual space of an algebra of observables is far too large to have physical importance. On the other hand, there is a class of states called Hadamard states which are considered to be physically sensible in many respects. The class is specified by fixing a common purely geometrical singularity structure, which defines the short distance behavior of n-point functions of a state. One merit of restricting to this class is the regularization of the 2-point functions by subtracting the common singular part. This allows to have well defined relative expectation values for the stress energy tensor. We hope to come to this topic in future. Another advantage is the conformity of the class of quasifree Hadamard states with the paradigm of the axiomatic locally covariant quantum field theory as mentioned earlier. The singularity structure was originally determined by Hadamard recurrent relations [25], from where it acquired its name. Later it was shown that this condition is equivalent to the so called microlocal spectral condition [48],[51], which is formulated in the language of wave front sets. We will not need the original definition in the current work, so we will immediately start with the newer formulation.

Our scope of fields is properly included in that of [51], so we will adopt some notations without special notice and refer to there for all definitions and results used in this context. However, in that paper one defines E to be anti-hermitian rather than antisymmetric, therefore the microlocal spectral condition looks reversed compared to the original form which we will also use. A bi-distribution $w \in (\mathcal{D}(\mathcal{T}) \otimes \mathcal{D}(\mathcal{T}))'$ is said to be of Hadamard form if

$$WF(w) = \{(q, k; q', k') \in \mathcal{N}_+ \times \mathcal{N}_- : (q, k) \sim (q', -k')\},$$

where $\mathcal{N}_\pm$ is the forward/backward null bundle, and $\sim$ means equality under a parallel transport along a null geodesic. We will say that a quasifree state ω is Hadamard if its 2-point function ω_2 is of Hadamard form.

In **Proposition 4.1** there is a striking asymmetry in the matrix $\hat{\omega}$ of a quasi-free state. We will separate this asymmetry by writing $\omega_2 = w_T + \omega_2^s$, where

$$\hat{\omega}^s = \begin{pmatrix} a^\omega & b^\omega \\ \bar{b}^\omega_{--} & \bar{a}^\omega_{--} \end{pmatrix}, \; \hat{w}_T = \begin{pmatrix} 0 & 0 \\ \delta & 0 \end{pmatrix}.$$

Both ω_2^s and w_T are weak bi-solutions, but ω_2^s is symmetric whereas w_T satisfies CCR,

$$\omega_T(f,g) - \omega_T(g,f) = -i\langle E[f], g\rangle_M.$$

If the fiber metric of $\mathcal{T}$ is Riemannian, then $s(\alpha) = 1$ everywhere, and w_T is positive. It defines an invariant quasifree pure state. But in general w_T is not positive, and does not define a state. In fact it does not contain state dependent information. It depends only on the geometry and the choice of modes T_α, hence we have put an index T. If we switch to the variable $s(t)$ and write $T_\alpha(s) = \rho_\alpha(s)e^{i\varphi_\alpha(s)}$ where $\rho_\alpha = |T_\alpha|$ and $\varphi_\alpha = \arg T_\alpha$, then the restriction $\det W[T_\alpha, \bar{T}_\alpha](s) = i$ forces $\dot{\varphi}_\alpha = \rho_\alpha^{-2}(s) \geq 0$. Therefore we will say that T_α is a positive frequency mode solution. Then $\bar{T}_\alpha$ is a negative frequency mode solution. Noting that $-i\langle E[f], g\rangle_M = w_T(f,g) - w_{\bar{T}}(f,g)$ we say that w_T is the positive frequency part of the propagator. In [20] it was shown that for the electromagnetic and Proca fields on an ultrastatic spacetime with compact spatial sections there exists a choice of modes T_α such that w_T has the Hadamard form. We will try to show that this is as well the case in our situation. We start by adapting the result of [20] to our setup. The mode decomposition makes the task much easier. The following trivial proposition will help to circumvent the need for compactness of the spatial sections, which [20] stipulated in order to rule out the vicinities of 0 from the spectrum of D_{Σ_t}.

Proposition 4.4 *If the bundle $\mathcal{T}$ is ultrastatic, i.e, $\Lambda_\alpha(s) = \Lambda_\alpha \geq 0$, then for any $f \in C_0^\infty(s(\mathcal{I}))$, $k_0 \in \mathbb{R}$, $N \in \mathbb{N}$,*

$$\Lambda_\alpha^N \int_{s(\mathcal{I})} ds f(s) T_\alpha(s) e^{ik_0 s} = \int_{s(\mathcal{I})} ds P_{2N}(k_0; s) T_\alpha(s) e^{ik_0 s},$$

where

$$P_{2N}(k_0; s) = (-1)^N \sum_{j=0}^{2N} C_{2N}^j f^{(j)}(s)(ik_0)^{2N-j}.$$

Proof: We only need to substitute $\Lambda_\alpha^N T_\alpha(s) = (-1)^N T_\alpha^{(2N)}(s)$ and to perform integration by parts $2N$ times. □

We will write $f(x) = \mathfrak{o}(|x|^{-\infty})$ to mean that f is of rapid decay in $|x| \to \infty$.

Proposition 4.5 *Suppose $\mathcal{T}$ is ultrastatic. Choose the initial data $T_\alpha(0) = p(\Lambda_\alpha)$ and*

$\dot{T}_\alpha(0) = q(\Lambda_\alpha)$ *such that*

$$p(\Lambda) - \frac{1}{\sqrt[4]{4\Lambda}} = \mathfrak{o}(\Lambda^{-\infty}),\ q(\Lambda) - i\sqrt[4]{\frac{\Lambda}{4}} = \mathfrak{o}(\Lambda^{-\infty}).$$

Then w_T *is of Hadamard form.*

Proof: First of all note that such a choice is possible, because the asymptotics of p and q satisfy the normalization condition. Because the microlocal condition is atlas independent, we will check the condition choosing the variable $s(t)$. Being a bi-solution of the field equation w_T satisfies $WF(w_T) \subset \mathcal{N} \times \mathcal{N}$. Choose any coordinate patches on regions $K, L \subset M$. Consider the restriction to $f(x) = f_0(s)f_1(\vec{x})$ and $g(y) = g_0(s')g_1(\vec{y})$ with $f_0, g_0 \in C_0^\infty(s(\mathcal{I}))$ and $f_1, g_1 \in \mathcal{D}(\mathcal{T}_t)$ so that $\mathrm{supp}\{f\} \subset K$ and $\mathrm{supp}\{g\} \subset L$. Then

$$w_T(fe^{ik*x}, ge^{ik'*y}) = \int_{\tilde{\Sigma}} d\mu(\alpha)s(\alpha)v_{-\alpha}(fe^{ik*x})u_\alpha(ge^{ik'*y}) =$$

$$= \int_{\tilde{\Sigma}} d\mu(\alpha)s(\alpha)\overline{u_\alpha(\bar{f}e^{-ik*x})}u_\alpha(ge^{ik'*y}) =$$

$$= \int_{\tilde{\Sigma}} d\mu(\alpha)s(\alpha)\overline{T_\alpha(\bar{f}_0e^{-ik_0s})[\widetilde{\bar{f}_1(\vec{x})e^{-\vec{k}\vec{x}}}](-\alpha)}T_\alpha(g_0e^{ik_0's'})[\widetilde{g_1(\vec{x})e^{\vec{k}\vec{x}}}](-\alpha).$$

Denote $\tilde{\Sigma}^{\leq 1} = \{\alpha \in \tilde{\Sigma}\colon\ \Lambda_\alpha \leq 1\}$. Choose $N \in \mathbb{N}$ arbitrarily. For $\alpha \in \tilde{\Sigma}^{\leq 1}$ write

$$T_\alpha(\bar{f}_0e^{-ik_0s}) = \int_{s(\mathcal{I})} dsT_\alpha(s)\bar{f}_0(s)e^{-ik_0s} = \frac{1}{(-ik_0)^N}\int_{s(\mathcal{I})} dse^{-ik_0s}\frac{d^N}{ds^N}T_\alpha(s)\bar{f}_0(s) \leq$$

$$\leq \frac{|\mathrm{supp}\{f_0\}|\sup_{\tilde{\Sigma}^{\leq 1}}\{|\partial_s^N T_\alpha f_0|\}}{1+|k_0|^N} \doteq \frac{c_N^1}{1+|k_0|^N},$$

where we used the fact that $T_\alpha(s)$ for any s depend only on Λ_α and is hence uniformly bounded along with all its derivatives on $\tilde{\Sigma}^{\leq 1}$. For $\alpha \in \tilde{\Sigma} \setminus \tilde{\Sigma}^{\leq 1}$ we have

$$T_\alpha(\bar{f}_0e^{-ik_0s}) = \int_{s(\mathcal{I})} ds\frac{1}{\sqrt[4]{4\Lambda_\alpha}}e^{i\sqrt{\Lambda_\alpha}s}\bar{f}_0(s)e^{-ik_0s} + \int_{s(\mathcal{I})} ds\left(T_\alpha(s) - \frac{1}{\sqrt[4]{4\Lambda_\alpha}}e^{i\sqrt{\Lambda_\alpha}s}\right)\bar{f}_0(s)e^{-ik_0s} =$$

$$= \frac{1}{\sqrt[4]{4\Lambda_\alpha}}\tilde{\bar{f}}_0(-\sqrt{\Lambda_\alpha}+k_0) + \int_{s(\mathcal{I})} ds\left(T_\alpha(s) - \frac{1}{\sqrt[4]{4\Lambda_\alpha}}e^{i\sqrt{\Lambda_\alpha}s}\right)\bar{f}_0(s)e^{-ik_0s}.$$

The ultrastatic spectrum is in particular loosely uniform, hence by **Proposition 1.4** we have

$$\sup_{\mathrm{supp}\{f_0\}}\left|T_\alpha(s) - \frac{1}{\sqrt[4]{4\Lambda_\alpha}}e^{i\sqrt{\Lambda_\alpha}s}\right| = \mathfrak{o}(\Lambda_\alpha^{-\infty}).$$

It follows $\exists c_N^2 \in \mathbb{R}_+$ with

$$(\Lambda_\alpha)^N \left| \int_{s(\mathcal{I})} ds \left(T_\alpha(s) - \frac{1}{\sqrt[4]{4\Lambda_\alpha}} e^{i\sqrt{\Lambda_\alpha} s} \right) \bar{f}_0(s) e^{-ik_0 s} \right| \leq c_N^2, \, \forall k_0 \in \mathbb{R}, \, \forall \alpha \in \tilde{\Sigma} \setminus \tilde{\Sigma}^{\leq 1}.$$

Combining with **Proposition 4.4** we find

$$\left| \int_{s(\mathcal{I})} ds \left(T_\alpha(s) - \frac{1}{\sqrt[4]{4\Lambda_\alpha}} e^{i\sqrt{\Lambda_\alpha} s} \right) P_{2N}(k_0; s) e^{-ik_0 s} \right| \leq c_N^2.$$

It is easy to see that

$$P_{2N}(k_0; s) = k_0^{2N} (f(s) + \frac{Q_{2N-1}(k_0; s)}{k_0^{2N}})$$

for some $Q_{2N-1}(k_0; s)$ which is of order $2N-1$ in k_0. Then

$$\left| \int_{s(\mathcal{I})} ds \left(T_\alpha(s) - \frac{1}{\sqrt[4]{4\Lambda_\alpha}} e^{i\sqrt{\Lambda_\alpha} s} \right) P_{2N}(k_0; s) e^{-ik_0 s} \right| \geq$$

$$\geq k_0^{2N} \left| \left| \int_{s(\mathcal{I})} ds \left(T_\alpha(s) - \frac{1}{\sqrt[4]{4\Lambda_\alpha}} e^{i\sqrt{\Lambda_\alpha} s} \right) f(s) e^{-ik_0 s} \right| - \right.$$

$$\left. - \left| \int_{s(\mathcal{I})} ds \left(T_\alpha(s) - \frac{1}{\sqrt[4]{4\Lambda_\alpha}} e^{i\sqrt{\Lambda_\alpha} s} \right) \frac{Q_{2N-1}(k_0; s)}{k_0^{2N}} e^{-ik_0 s} \right| \right|.$$

For sufficiently large $|k_0|$ we have

$$\left| \int_{s(\mathcal{I})} ds \left(T_\alpha(s) - \frac{1}{\sqrt[4]{4\Lambda_\alpha}} e^{i\sqrt{\Lambda_\alpha} s} \right) \frac{Q_{2N-1}(k_0; s)}{k_0^{2N}} e^{-ik_0 s} \right| \leq$$

$$\frac{1}{2} \left| \int_{s(\mathcal{I})} ds \left(T_\alpha(s) - \frac{1}{\sqrt[4]{4\Lambda_\alpha}} e^{i\sqrt{\Lambda_\alpha} s} \right) f(s) e^{-ik_0 s} \right|,$$

thereby we can find $0 < c_N^3 \in \mathbb{R}$ such that

$$\left| \int_{s(\mathcal{I})} ds \left(T_\alpha(s) - \frac{1}{\sqrt[4]{4\Lambda_\alpha}} e^{i\sqrt{\Lambda_\alpha} s} \right) f(s) e^{-ik_0 s} \right| \leq \frac{c_N^3}{1 + |k_0|^N}.$$

Finally look at the term

$$\frac{1}{\sqrt[4]{4\Lambda_\alpha}} \tilde{\bar{f}}_0(-\sqrt{\Lambda_\alpha} + k_0).$$

Because $f_0 \in C_0^\infty(s(\mathcal{I}))$ we know that $\hat{f}(k) = \mathfrak{o}(|k|^{-\infty})$. If $k_0 \to -\infty$, then the expression obviously decays rapidly in $|k_0|$ uniformly in α. But if $k_0 \to +\infty$ there is always an α

with $\sqrt{\Lambda_\alpha} = k_0$ and thus

$$\frac{1}{\sqrt[4]{4\Lambda_\alpha}}\tilde{\bar{f}}_0(-\sqrt{\Lambda_\alpha}+k_0) = \frac{1}{\sqrt[4]{4k_0}}\tilde{\bar{f}}_0(0),$$

therefore the expression is not of uniform rapid decay. Summarizing all this we establish that the expression $T_\alpha(\bar{f}_0 e^{-ik_0 s})$ is uniformly of rapid decay in $|k_0|$ if and only if $k_0 < 0$. Similar arguments can be applied to $T_\alpha(g_0 e^{ik_0' s'})$ with the same conclusion on $k_0' > 0$. Combining these two we find that $WF(w_T) \subset \mathcal{N}_+ \times \mathcal{N}_-$. Now we apply the reasoning of [20]. It was shown in [51] that there exists a Hadamard bi-distribution w_0 which satisfies CCR. Then $WF(w_0) \subset \mathcal{N}_+ \times \mathcal{N}_-$, and thereby $WF(w_T - w_0) \subset \mathcal{N}_+ \times \mathcal{N}_-$. But both w_T and w_0 satisfy CCR, and therefore $w_T - w_0$ is a symmetric distribution. A symmetric distribution has a symmetric wave front set, hence $WF(w_T - w_0) \subset (\mathcal{N}_+ \times \mathcal{N}_-) \cap (\mathcal{N}_- \times \mathcal{N}_+) = \emptyset$. Thus $w_T - w_0$ is smooth, and it follows that w_T is of Hadamard form. □

Now we come back to our generic bundle $\mathcal{T}$ and see how we can obtain Hadamard regularizers from those on ultrastatic bundles.

Proposition 4.6 *There exist a choice of mode solutions T_α such that w_T is of Hadamard form.*

Proof: By local-to-global theorem it suffices to show that w_T is locally Hadamard. Further, by the propagation of singularities theorem we need only to show that w_T is locally Hadamard in a causal normal neighborhood of a Cauchy surface (see [51] for both theorems). Let $\mathcal{T}^{us}$ be an ultrastatic bundle homeomorphic to $\mathcal{T}$ with spacetime metric g^{us}, fiber metric $\mathfrak{g}^{us}$, connection form Γ^{us} and the mass term m^{us}. Choose a non-negative function $f \in C_0^\infty([0,1])$ and let $F(t) = \int_0^t d\tau f(\tau)$. Define the 'welded' bundle $\mathcal{T}^w$ homeomorphic to $\mathcal{T}$ with the spacetime metric $g^w = g^{us}F(t) + g(1-F(t))$, fiber metric $\mathfrak{g}^w = \mathfrak{g}^{us}F(t) + \mathfrak{g}(1-F(t))$, connection form $\Gamma^w = \Gamma^{us}F(t) + \Gamma(1-F(t))$ and mass term $m^w = m^{us}F(t) + m(1-F(t))$. Now choose the modes T_α^{us} according to **Proposition 4.5** so that w_T^{us} is Hadamard on $\mathcal{T}^{us}$. Then, in particular, w_T^{us} will be locally Hadamard in the portion $t > 1$ of $\mathcal{T}^{us}$. Define the modes T_α^w on $\mathcal{T}^w$ by $T_\alpha^w(1) = T_\alpha^{us}(1)$ and $\dot{T}_\alpha^w(1) = \dot{T}_\alpha^{us}(1)$. Because the portion $t > 1$ of $\mathcal{T}^{us}$ is isomorphic to the same portion $t > 1$ of $\mathcal{T}^w$, with the modes T_α^w the bi-distribution w_T^w will be locally Hadamard for $t > 1$. But then w_T^w will be also globally Hadamard on entire $\mathcal{T}^w$, and in particular, locally Hadamard in the portion $t < 0$ of it. Finally we let the modes T_α of our original bundle $\mathcal{T}$ be defined by the initial data $T_\alpha(0) = T_\alpha^w(0)$ and $\dot{T}_\alpha(0) = \dot{T}_\alpha^w(0)$. Now by the same reasoning w_T is locally Hadamard for $t < 0$ in $\mathcal{T}$, and hence globally Hadamard in $\mathcal{T}$. $\square$

Note that though the proof is not manifestly constructive, the desired modes can be easily computed numerically. What one needs is to freeze the functions Λ_α on an interval $[a, b]$ to some constant values with a suitable mollifier f, then choose initial data $T_\alpha(a)$, $\dot{T}_\alpha(a)$ according to **Proposition 4.5** with respect to these frozen values, let the modes evolve back to 0 and pick the data $T_\alpha(0)$, $\dot{T}_\alpha(0)$. One can think of these desired initial data as the evaluation of a special function at 0. Of course, there are more explicit constructions of Hadamard regularizers in the literature. For instance, the states of low energy found in [42]. But the techniques involved there include a large amount of manipulations with the so called adiabatic vacua, which we would like to avoid. Besides, those results have been obtained so far only for the Klein-Gordon field on FRW spacetimes. Another approach using the so called boundary-to-bulk correspondence has been performed recently in [13] for asymptotically flat spacetimes. However, it is not immediately clear how one can relate this to the setup of the mode decomposition.

Remark 4.3 *If T_α is chosen such that w_T is Hadamard, then a quasifree state ω is Hadamard if and only if the bi-distribution ω_2^s given by the matrix $\hat{\omega}^s$ is smooth. Because w_T is state independent, we can think of ω_2^s as the Hadamard regularized 2-point function $:\omega_2:$.*

Now if the quasifree state ω is given by the coefficient distributions a^ω, b^ω, it is not at once obvious whether ω_2^s is smooth or not. This is an intricate problem in harmonic analysis to find the Fourier analog of the smoothness condition on a distribution. If the distribution is given by a kernel which is L^2 along with all its derivatives, then we have found the answer in chapter 2. The general case is a point of ongoing discussion with an expert in the field S. Thangavelu, and we hope to be able to announce positive results in the future.

Chapter 5

Appendix

5.1 Space structures. Distributions

Let us start with introducing symmetric metric products

$$\langle f,h\rangle_M = \int_M d\mu_g(x)\langle f(x),h(x)\rangle_g,\; f\in\mathcal{E}(\mathcal{T}),\; h\in\mathcal{D}(\mathcal{T}),$$

$$\langle f,h\rangle_{\Sigma_t} = \int_\Sigma d\mu_h(\vec{x})\langle f(\vec{x}),h(\vec{x})\rangle_g,\; f\in\mathcal{E}(\mathcal{T}_t),\; h\in\mathcal{D}(\mathcal{T}_t).$$

The pseudo-Riemannian metric $\langle,\rangle_\mathfrak{g}$ induces a Krein space structure on V, the typical fiber of $\mathcal{T}$. Whence there is a Krein involution $\check{\Gamma}$, such that $(u,v)_\mathfrak{g} = \langle\bar{u},\check{\Gamma}v\rangle_\mathfrak{g}$, $u,v\in V$, is a positive definite hermitian inner product. This gives rise to positive definite hermitian inner products

$$(f,h)_M = \int_M d\mu_g(x)(f(x),h(x))_\mathfrak{g},\; f\in\mathcal{E}(\mathcal{T}),\; h\in\mathcal{D}(\mathcal{T}),$$

$$(f,h)_{\Sigma_t} = \int_\Sigma d\mu_h(\vec{x})(f(\vec{x}),h(\vec{x}))_\mathfrak{g},\; f\in\mathcal{E}(\mathcal{T}_t),\; h\in\mathcal{D}(\mathcal{T}_t).$$

The completion of spaces $\mathcal{D}(\mathcal{T})$ and $\mathcal{D}(\mathcal{T}_t)$ with respect to these products becomes the Hilbert spaces $L^2(\mathcal{T})$ and $L^2(\mathcal{T}_t)$, respectively. The tangent space T_pM at a point $p\in M$ with the Lorentzian metric g is another example of a Krein space. In the same spirit one defines the positive definite inner product $(,)_g$ on T_pM. The metric h on $T_p\Sigma$ is

Riemannian, so the construction of $(,)_h$ is straightforward. Note that $\langle,\rangle_{\mathfrak{g}}$ and g together give pseudo-Riemannian metrics on all product bundles $T^*M\otimes...\otimes T^*M\otimes\mathcal{T}$ (respectively, $\langle,\rangle_{\mathfrak{g}}$ and h on $T^*\Sigma\otimes...\otimes T^*\Sigma\otimes\mathcal{T}_t$). All the resulting standard fibers are again Krein spaces, and can be given inner products $(,)_g$ in the same fashion. These in their turn produce products $(,)_M$ and $(,)_{\Sigma_t}$ on the respective sections.

The perfect countably Banach topology of the test function spaces $\mathcal{D}(\mathcal{T})$ and $\mathcal{D}(\mathcal{T}_t)$ can be given as usual (f.i.,[2]). However, as we are going to perform a spectral analysis, we will need nuclear countably Hilbert space structure, to which we proceed [38]. Let $\mathcal{O}\subset M$ be a compact region. Let

$$\mathcal{D}_{\mathcal{O}}(\mathcal{T})=\{f\in\mathcal{D}(\mathcal{T})\colon\ supp\{f\}\subset\mathcal{O}\}$$

and define the family of positive definite inner products $(,)_{\mathcal{O},p}$ on $\mathcal{D}_{\mathcal{O}}(\mathcal{T})$ by

$$(f,h)_{\mathcal{O},p}=\sum_{q\leq p}((\nabla)^q f,(\nabla)^q h)_M,\ \forall f,h\in\mathcal{D}_{\mathcal{O}}(\mathcal{T}),\ p,q\in\mathbb{N},$$

which induces a family of norms $\|.\|_{\mathcal{O},p}$. One can show that this family of norms is growing and consistent, and gives the same topology as the usual one. Let us give $\mathcal{D}_{\mathcal{O}}(\mathcal{T})$ a countably Hilbert space structure in the following sense,

$$\mathcal{D}_{\mathcal{O}}(\mathcal{T})=\bigcap_{\mathbb{N}}\overline{\mathcal{D}_{\mathcal{O}}(\mathcal{T})}^{(,)_{\mathcal{O},p}}.$$

It can be shown, that thus constructed countably Hilbert space $\mathcal{D}_{\mathcal{O}}(\mathcal{T})$ is nuclear. Let now

$$\mathcal{O}_1\subset...\subset\mathcal{O}_n\subset...\subset M$$

be an infinite family of growing compact regions. Then give $\mathcal{D}(\mathcal{T})$ the inductive limit topology

$$\mathcal{D}(\mathcal{T})=\lim_{n\to\infty}\mathcal{D}_{\mathcal{O}_n}(\mathcal{T}).$$

Here we are done. Distributions $\mathcal{D}(\mathcal{T})'$ and operations on them can be defined as usual. The same construction can be done for $\mathcal{D}(\mathcal{T}_t)$ with minor modifications.

At the end let us consider the choice of the topology in detail. In the literature one usually chooses the family of norms $\|.\|_p$ (or sometimes a family of seminorms $|(.)|_p$; from these seminorms one can make norms by $\|.\|_p=\sum_{q<p}|(.)|_q$ or $\|.\|_p=\sup_{q<p}|(.)|_q$ etc.) rather arbitrarily in accordance with the setup of the problem, and it is tacitly assumed

but not everywhere proven, that all such choices give equivalent topologies. Let us for consistency present here a proof of this fact. The zest of the proof (the usage of the Sobolev embedding theorem) was suggested by G. Folland.

Proposition 5.1 *Let $\mathcal{T} \stackrel{\pi}{\to} M$ be an n dimensional pseudo-Riemannian vector bundle over the d-dimensional parallelizable pseudo-Riemannian manifold M with positive metric product $(,)_{\mathfrak{g}}$ constructed as above, so that we have well defined L^m norms $|||.|||_m$ for $1 \le m \le \infty$ on $\mathcal{D}(\mathcal{T})$. Let ∇ be a connection on $\mathcal{T}$. Suppose that*

(i) $X_1...X_d$ be a system of first order smooth differential operators on $C^\infty(\mathcal{T})$ which span the tangent space T^*M everywhere,
(ii) the seminorms be given by $\|(f)\|_{\alpha,q} = |||P_{\alpha,q}(X_i)f|||_m$, where $P_{\alpha,q}(X_i)$ are various monomials of order q in $\{X_i\}$, $f \in \mathcal{D}(\mathcal{T})$,
(iii) the family of norms be given as $\|f\|_p = |||\{\|(f)\|_{\alpha,q}\}_{q\le p}|||_{l^k}$,
or by a superposition of different $|||.|||_{l^k}$, $1 \le k \le \infty$. (5.1)

Then the topology of $\mathcal{D}(\mathcal{T})$ defined by this family of norms is independent of the decisions (i) to (iii).

Proof: For convenience denote by $(X_i, m, *)$ the triple of choices at points (i),(ii) and (iii). Then $(X_i, m, *) \sim (X_i', m', *')$ will mean that this two topologies are equivalent.

As the topology of $\mathcal{D}(\mathcal{T})$ is the inductive limit of various $\mathcal{D}(\mathcal{T}_K)$ with $\mathcal{T}_K = \pi^{-1}(K)$, $K \subset M$ compact, it suffices to prove the assertion for an arbitrary $\mathcal{D}(\mathcal{T}_K)$. The topologies given by two families of norms $\{\|.\|_p\}$ and $\{\|.\|_p'\}$ are equivalent if and only if these two systems of norms are themselves equivalent, i.e., $\forall p$, $\exists q(p), r(p) > 0$, $0 < C_p, C_p' \in \mathbb{R}$ such that $\|.\|_p \le C_p\|.\|_{q(p)}$ and $\|.\|_p' \le C_p'\|.\|_{r(p)}'$. Let us start with the point (iii). Suppose the choices (i) and (ii) are fixed, i.e., consider $(X_i, m, *)$ and $(X_i, m, *')$. Then all possible choices in (iii) give equivalent systems of norms because of the elementary inequalities

$$|||\{\|(f)\|_{\alpha,q}\}_I|||_{l^\infty} \le ... \le |||\{\|(f)\|_{\alpha,q}\}_I|||_{l^k} \le ... \le |||\{\|(f)\|_{\alpha,q}\}_I|||_{l^1} \le N_I|||\{\|(f)\|_{\alpha,q}\}_I|||_{l^\infty},$$

where N_I is the number of terms in the index set I. These inequalities can be applied consecutively to estimate any composite norm by, say, $|||.|||_{l^\infty}$. An example of a composite norm is $\|f\|_p = \sup_{q\le p}|||\nabla^q f|||_\infty$. We found that $(X_i, m, *) \sim (X_i, m, *')$.

Now let $1 \leq m \leq \infty$ at (ii) and $k = \infty$ at (iii) be chosen, and choose two systems of operators $\{X_i\}$ and $\{Y_i\}$ at point (i) to construct the families of norms $\{\|.\|_p\}$ and $\{\|.\|'_p\}$, respectively. This corresponds to (X_i, m, l^∞) and (Y_i, m, l^∞). Because $\{X_i\}$ spans T^*M, there are functions $c_{ij}(x) \in C^\infty(M)$ and smooth fields of homomorphisms $\tilde{\Gamma}_i \in C^\infty(Hom(\mathcal{T}, \mathcal{T}))$ with $Y_i(x) = \sum_j c_{ij}(x) X_j(x) + \tilde{\Gamma}_i$. Using this for any monomial $P_{\alpha,q}(Y_i)$ we get

$$P_{\alpha,q}(Y_i) f = \sum_\beta c^\beta_{\alpha,q}(x) Q^\beta_{\alpha,q}(X_i) f,$$

where $c^\beta_{\alpha,q}(x) \in C^\infty(M)$ and $Q^\beta_{\alpha,q}(X_i)$ are monomials of order less or equal q. The number of summands is less than, say, $(4d)^q$. It follows by Minkowsky inequality

$$|(f)|'_{\alpha,q} = |||P_{\alpha,q}(Y_i) f|||_m \leq \sum_\beta |||c^\beta_{\alpha,q}(x) Q^\beta_{\alpha,q}(X_i) f|||_m,$$

and then by Hoelder inequality

$$\sum_\beta |||c^\beta_{\alpha,q}(x) Q^\beta_{\alpha,q}(X_i) f|||_m \leq C_{\alpha,q} \sum_\beta |||Q^\beta_{\alpha,q}(X_i) f|||_m = C_{\alpha,q} \sum_\beta |(f)|_{\alpha(\alpha,q,\beta), q(\alpha,q,\beta)},$$

where $0 < C_{\alpha,q} = \sup_\beta |||c^\beta_{\alpha,q}|||_\infty$. In other words, the seminorms of order q of the second system can be estimated by linear combinations of seminorms of the first system of the same or lower order. Then

$$\|f\|'_p = \sup_{q \leq p} |(f)|'_{\alpha,q} \leq C_p \sup_{q \leq p} \sum_\beta |(f)|_{\alpha(\alpha,q,\beta), q(\alpha,q,\beta)} \leq$$

$$\leq C_p (4d)^p \sup_{q \leq p} |(f)|_{\alpha(\alpha,q,\beta), q(\alpha,q,\beta)} \leq C_p (4d)^p \sup_{q \leq p} |(f)|_{\alpha,q} = C_p (4d)^p \|f\|_p,$$

where $0 < C_p = \sup_{q \leq p} C_{\alpha,q}$. For the other direction of the estimate we simply need to switch $\{X_i\}$ and $\{Y_i\}$. Thus these two topologies are equivalent, $(X_i, m, l^\infty) \sim (Y_i, m, l^\infty)$.

Finally let $X_i = \nabla_i$ (components with respect to a global orthonromal frame in T^*M) be chosen at (i), and $\|.\| = |||\{|(.)|_{\alpha,q}\}_{q \leq p}|||_{l^2}$ at (iii). We construct two families of norms by choosing $1 \leq m < \infty$ and $m' = \infty$ at (ii) for $\|.\|_p$ and $\|.\|'_p$, respectively. This can be symbolized as (∇_i, m, l^2) and (∇_i, ∞, l^2). Because K is compact, by an application of Hoelder inequality we obtain

$$|||.|||_m \leq C_m |||.|||_\infty$$

for some $0 < C_m \in \mathbb{R}$, and hence obviously

$$\|.\|_p \leq C_m \|.\|'_p,\ p \in \mathbb{N}_0.$$

The opposite inequality requires an application of Sobolev embedding theorem for compact manifolds [28],[55]. Denote the Sobolev norms (which are equivalent to those in [28])

$$|||f|||_{W^{p,m}} = \sqrt{\sum_{q \leq p} |||\nabla^q f|||_m^2}.$$

Then an application of Sobolev embedding theorem gives

$$|||.|||_{W^{0,\infty}} = |||.|||_\infty \leq D|||.|||_{W^{d,1}}$$

for some $0 < D \in \mathbb{R}$. By another application of Hoelder inequality we find

$$|||.|||_{W^{d,1}} \leq |||.|||_{W^{d,2}},$$

and therefore

$$|||.|||_\infty \leq D\sqrt{\sum_{q \leq d} |||\nabla^q f|||_2^2}.$$

Next

$$|||\nabla^q f|||_2^2 = \sum_\alpha |||P_{\alpha,q}(X_i) f|||_2^2,$$

and finally

$$\|f\|'_p = \sqrt{\sum_{q \leq p} |||P_{\alpha,q}(X_i) f|||_\infty^2} \leq D\sqrt{\sum_{q \leq p} \sum_{j \leq d} |||\nabla^j P_{\alpha,q}(X_i) f|||_2^2} =$$

$$= D\sqrt{\sum_{q \leq p} \sum_{j \leq d} |||P_{\beta,j}(X_i) P_{\alpha,q}(X_i) f|||_2^2} \leq D\sqrt{\sum_{q \leq p+d} |||P_{\alpha,q}(X_i) f|||_2^2} \leq$$

$$\leq DE\sqrt{\sum_{q \leq p+d} |||P_{\alpha,q}(X_i) f|||_m^2} = DE\|f\|'_{p+d},$$

where in the last inequality again Hoelders inequality was used with some $0 < E \in \mathbb{R}$. Thus we have shown that choosing any $1 \leq m < \infty$ is equivalent to choosing $m = \infty$ at point (ii), i.e., $(\nabla_i, m, l^2) \sim (\nabla_i, \infty, l^2)$.

Write

$$(X_i, m, *) \sim (X_i, m, l^\infty) \sim (\nabla_i, m, l^\infty) \sim (\nabla_i, m, l^2) \sim (\nabla_i, \infty, l^2) \sim (\nabla_i, m', l^2) \sim$$

$$\sim (\nabla_i, m', l^\infty) \sim (X'_i, m', l^\infty) \sim (X'_i, m', *').$$

The proof is complete. □

5.2 On the time dependent harmonic oscillator

Here we will concentrate on some properties of the solutions of the smooth complex time dependent harmonic oscillator equation

$$\ddot{T}(s) + \Lambda(s)T(s) = 0 \tag{5.2}$$

where $\Lambda(s)$ is a smooth complex function on the real line. This equation is under attention since a long time, but some results are not that easily available today (at least for us).

We start with an easy remark. Denote by

$$W[Q,R](s) = \begin{pmatrix} Q(s) & \dot{Q}(s) \\ R(s) & \dot{R}(s) \end{pmatrix}$$

the Wronski matrix of two solutions Q and R.

Remark 5.1 *Let Q, R be two linearly independent solutions of (Eq.5.2), and T an arbitrary solution. Then from the conservation of* $\det W[Q,T]$ *and* $\det W[R,T]$ *it is easy to find*

$$\begin{pmatrix} \dot{T}(s) \\ -T(s) \end{pmatrix} = W[Q,R]^{-1}(s) \times W[Q,R](0) \times \begin{pmatrix} \dot{T}(0) \\ -T(0) \end{pmatrix} =$$

$$= \det W[Q,R]^{-1}(0) \begin{pmatrix} \dot{R}(s) & -\dot{Q}(s) \\ -R(s) & Q(s) \end{pmatrix} \times \begin{pmatrix} Q(0) & \dot{Q}(0) \\ R(0) & \dot{R}(0) \end{pmatrix} \times \begin{pmatrix} \dot{T}(0) \\ -T(0) \end{pmatrix}.$$

Thus having at hand two such particular solutions Q, R, we have a control over arbitrary solutions T in terms of their initial data.

Our first task is to obtain a control over the magnitude of the solution T on a given compact interval $\mathcal{R}$ in terms of its initial data $T(0)$ and $\dot{T}(0)$. This is done by the so called energy estimate. Define the energy of a solution T by

$$\mathcal{W}[\breve{T}](s) = \frac{1}{2}|\dot{T}|^2(s) + \frac{1}{2}\Re\Lambda(s)|T(s)|^2.$$

If $\Re\Lambda > 0$ on $\mathcal{R}$ then $2\mathcal{W}[\breve{T}]$ dominates $\Re\Lambda|T|^2$ and $|\dot{T}|^2$, and obtaining bounds on $\mathcal{W}[\breve{T}]$ we automatically get bounds on $|T|$ and $|\dot{T}|$.

Proposition 5.2 *For arbitrary solution T of*

$$\ddot{T}(s) + \Lambda(s)T(s) = 0,$$

with smooth complex valued $\Lambda(s)$ having a positive real part (i.e., $\Re\Lambda(s) > 0$) on a compact interval $\mathcal{R}$, the energy function $\mathcal{W}[\breve{T}](s)$ satisfies the estimate

$$\mathcal{W}[\breve{T}](0)e^{-\int_0^s d\sigma\left(\frac{2|\Im\Lambda(\sigma)|}{\sqrt{\Re\Lambda(\sigma)}}+|\partial_s \ln \Re\Lambda(\sigma)|\right)} \leq \mathcal{W}[\breve{T}](s) \leq \mathcal{W}[\breve{T}](0)e^{\int_0^s d\sigma\left(\frac{2|\Im\Lambda(\sigma)|}{\sqrt{\Re\Lambda(\sigma)}}+|\partial_s \ln \Re\Lambda(\sigma)|\right)}$$

for all $s \in \mathcal{R}$.

Proof: Write $T(s) = R(s) + iS(s)$, $\Lambda(s) = \Theta(s) + i\Xi(s)$, and insert into the equation. We will get the following system of real equations,

$$\begin{cases} \ddot{R}(s) + \Theta(s)R(s) - \Xi(s)S(s) = 0, \\ \ddot{S}(s) + \Theta(s)S(s) + \Xi(s)R(s) = 0. \end{cases}$$

We can cast this into a real vector equation

$$\ddot{\breve{T}}(s) + \hat{\Lambda}(s)\breve{T}(s) = 0$$

by denoting

$$\breve{T}(s) = (R(s), S(s))^\top,$$

and

$$\hat{\Lambda}(s) = \begin{pmatrix} \Theta(s) & -\Xi(s) \\ \Xi(s) & \Theta(s) \end{pmatrix} = \hat{\Lambda}^+(s) + \hat{\Lambda}^-(s) = \Theta(s)\mathbf{1} + \Xi(s)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where $\hat{\Lambda}^{\pm}$ denote the symmetric and antisymmetric parts. The energy function equals

$$\mathcal{W}[\breve{T}](s)=\frac{1}{2}\dot{\breve{T}}^2(s)+\frac{1}{2}\breve{T}^{\top}(s)\hat{\Lambda}(s)\breve{T}(s)=\frac{1}{2}\dot{\breve{T}}^2(s)+\frac{1}{2}\breve{T}^{\top}(s)\hat{\Lambda}^{+}(s)\breve{T}(s).$$

On the interval $\mathcal{R}$ we have $\mathcal{W}[\breve{T}](s)>0$ as by the assumption $\Theta(s)>0$. One can easily find that

$$\dot{\mathcal{W}}[\breve{T}](s)=\breve{T}^{\top}(s)\hat{\Lambda}^{-}(s)\dot{\breve{T}}(s)+\frac{1}{2}\breve{T}^{\top}(s)\dot{\hat{\Lambda}}^{+}(s)\breve{T}(s),$$

whence it follows

$$\left|\dot{\mathcal{W}}[\breve{T}](s)\right|\leq|\Xi(s)||\breve{T}(s)||\dot{\breve{T}}(s)|+|\partial_s\ln\Theta(s)|\mathcal{W}[\breve{T}](s).$$

By definition of $\mathcal{W}[\breve{T}]$ and positivity of Θ we have $|\dot{\breve{T}}(s)|\leq\sqrt{2\mathcal{W}[\breve{T}](s)}$ and $|\breve{T}(s)|\leq\sqrt{2\mathcal{W}[\breve{T}](s)/\Theta(s)}$ on $\mathcal{R}$. It follows then

$$\left|\partial_s\ln\mathcal{W}[\breve{T}](s)\right|\leq\frac{2|\Xi(s)|}{\sqrt{\Theta(s)}}+|\partial_s\ln\Theta(s)|,$$

and integrating this we finally arrive at

$$\mathcal{W}[\breve{T}](0)e^{-\left|\int_0^s d\sigma\left(\frac{2|\Xi(\sigma)|}{\sqrt{\Theta(\sigma)}}+|\partial_s\ln\Theta(\sigma)|\right)\right|}\leq\mathcal{W}[\breve{T}](s)\leq\mathcal{W}[\breve{T}](0)e^{\left|\int_0^s d\sigma\left(\frac{2|\Xi(\sigma)|}{\sqrt{\Theta(\sigma)}}+|\partial_s\ln\Theta(\sigma)|\right)\right|},$$

precisely as in the statement. □

If however Λ is not guaranteed to be positive, then on those regions where it is negative the magnitude of the solutions is expected to behave exponentially. We are able to capture that exponential factor by the following beautiful trick.

Proposition 5.3 *For any $0<\kappa\in\mathbb{R}$, any solution of the equation*

$$\ddot{T}(s)+\Lambda(s)T(s)=0$$

can be represented as $T(s)=\tau(\frac{1}{\kappa}\operatorname{th}(\kappa s))\operatorname{ch}(\kappa s)$, where $\tau(z)$ is a solution of the equation

$$\ddot{\tau}(z)+\Omega(z)\tau(z)=0$$

with

$$\Omega(z)=\frac{\kappa^2+\Lambda(\frac{1}{\kappa}\operatorname{ath}(\kappa z))}{(1-\kappa^2z^2)^2},\ z\in(-\frac{1}{\kappa},\frac{1}{\kappa}).$$

Proof: The proof is elementary once we already know the clue: the substitution of variables $\kappa z = \operatorname{th}(\kappa s)$. The substitution $T(s) = \tau(s)\operatorname{ch}(\kappa s)$ into the original equation gives

$$\ddot{\tau}(s) + 2\kappa \operatorname{th}(\kappa s)\dot{\tau}(s) + (\kappa^2 + \Lambda(s))\tau(s) = 0,$$

then the substitution $s \to z$ yields the final formulas. □

Let us say a couple of words about this. If $\Re\Lambda$ has a minimal negative value $-c$ on some domain, then it suffices to set $\kappa = \sqrt{c}$ to reduce the problem to an oscillatory equation for ρ. The upper bound of the rate of exponential expansion is precisely given by the square root of the minimal negative value of $\Re\Lambda$.

Finally we combine these two statements to find an explicit uniform bound on an arbitrary solution T. Let the compact interval $\mathcal{R}$ containing 0 be fixed, and set

$$A_{\mathcal{R}} = \sup_{\mathcal{R}} |\Im\Lambda|,\ c_{\mathcal{R}} = \inf_{\mathcal{R}} \Re\Lambda,\ \kappa = \sqrt{1 + |\min\{0, c_{\mathcal{R}}\}|},\ B_{\mathcal{R}} = \sup_{\mathcal{R}} \left|\partial_s \ln\left(\kappa^2 + \Re\Lambda\right)\right|,$$

$$D_{\mathcal{R}} = \sup_{\mathcal{R}}(\kappa^2 + \Re\Lambda),\ e_{\mathcal{R}} = \inf_{\mathcal{R}}(\kappa^2 + \Re\Lambda) = 1 + \max\{0, c_{\mathcal{R}}\},$$

$$L_{\mathcal{R}} = \left(2A_{\mathcal{R}} + \operatorname{ch}^2(\kappa|\mathcal{R}|)B_{\mathcal{R}} + 2\kappa \operatorname{sh}(2\kappa|\mathcal{R}|)\right)$$

(we suppressed the index $\mathcal{R}$ of κ for convenience).

Corollary 5.1 *For an arbitrary solution T it holds*

$$|T(s)| \leq |T(0)|\sqrt{\frac{D_{\mathcal{R}}}{e_{\mathcal{R}}}} e^{L_{\mathcal{R}}} \operatorname{ch}(\kappa|\mathcal{R}|) + |\dot{T}(0)| \frac{1}{\sqrt{e_{\mathcal{R}}}} e^{L_{\mathcal{R}}} \operatorname{ch}(\kappa|\mathcal{R}|)$$

for all $s \in \mathcal{R}$.

Proof: Consider the linearly independent solutions Q and R given by initial data

$$Q(0) = 1,\ \dot{Q}(0) = 0,\ R(0) = 0,\ \dot{R}(0) = 1.$$

Using **Proposition 5.3** represent them as $Q(s) = \xi(z(s))\operatorname{ch}(\kappa s)$ and $R(s) = \rho(z(s))\operatorname{ch}(\kappa s)$, where $\xi(z)$ and $\rho(z)$ are solutions of the equation

$$\ddot{\tau}(z) + \Omega(z)\tau(z) = 0$$

with

$$\Omega(z) = \frac{\kappa^2 + \Lambda(\frac{1}{\kappa}\operatorname{ath}(\kappa z))}{(1-\kappa^2 z^2)^2}.$$

Using $z(0) = 0$ and

$$\frac{d}{ds}\left[\tau(z(s))\operatorname{ch}(\kappa s)\right] = \frac{\dot{\tau}(z(s))}{\operatorname{ch}(\kappa s)} + \tau(z(s))\operatorname{sh}(\kappa s)\kappa, \tag{5.3}$$

we find

$$\xi(0) = 1,\ \dot{\xi}(0) = 0,\ \rho(0) = 0,\ \dot{\rho}(0) = 1.$$

Note that $\Re\Omega(s) = \kappa^2 + \Re\Lambda \geq 1$, thus **Proposition 5.2** is applicable for ξ and ρ. We have $\mathcal{W}[\xi](0) = \frac{1}{2}\Re\Omega(0)$ and $\mathcal{W}[\rho](0) = \frac{1}{2}$. Now

$$\frac{d}{dz}\ln\Re\Omega(z) = \frac{ds}{dz}(s)\frac{d}{ds}\ln\Re\Omega(z(s)) = \operatorname{ch}^2(\kappa s)\frac{d}{ds}\ln\left((\kappa^2 + \Re\Lambda(s))\operatorname{ch}^4(\kappa s)\right) =$$

$$= \operatorname{ch}^2(\kappa s)\frac{d}{ds}\ln\left(\kappa^2 + \Re\Lambda(s)\right) + 2\kappa\operatorname{sh}(2\kappa s).$$

Then it follows

$$\left|\int_0^z d\sigma\left(\frac{2|\Im\Omega(\sigma)|}{\sqrt{\Re\Omega(\sigma)}} + |\partial_z \ln\Re\Omega(\sigma)|\right)\right| \leq \frac{2}{\kappa}\operatorname{th}(\kappa|\mathcal{R}|)\left(2A_{\mathcal{R}} + \operatorname{ch}^2(\kappa|\mathcal{R}|)B_{\mathcal{R}} + 2\kappa\operatorname{sh}(2\kappa|\mathcal{R}|)\right) \leq$$

$$\leq 2\left(2A_{\mathcal{R}} + \operatorname{ch}^2(\kappa|\mathcal{R}|)B_{\mathcal{R}} + 2\kappa\operatorname{sh}(2\kappa|\mathcal{R}|)\right) = 2L_{\mathcal{R}}.$$

By **Proposition 5.2** we have

$$\mathcal{W}[\xi](z) \leq \mathcal{W}[\xi](0)e^{2L_{\mathcal{R}}},\ \mathcal{W}[\rho](z) \leq \mathcal{W}[\rho](0)e^{2L_{\mathcal{R}}},$$

which entails

$$|\xi(z)| \leq \sqrt{\frac{\kappa^2 + \Re\Lambda(0)}{\kappa^2 + \Re\Lambda(s(z))}}e^{L_{\mathcal{R}}},\ |\dot{\xi}(z)| \leq \sqrt{\kappa^2 + \Re\Lambda(0)}e^{L_{\mathcal{R}}},$$

$$|\rho(z)| \leq \frac{1}{\sqrt{\kappa^2 + \Re\Lambda(s(z))}}e^{L_{\mathcal{R}}},\ |\dot{\rho}(z)| \leq e^{L_{\mathcal{R}}}.$$

For Q and R we get

$$|Q(s)| \leq \sqrt{\frac{D_{\mathcal{R}}}{e_{\mathcal{R}}}}e^{L_{\mathcal{R}}}\operatorname{ch}(\kappa|\mathcal{R}|),\ |R(s)| \leq \frac{1}{\sqrt{e_{\mathcal{R}}}}e^{L_{\mathcal{R}}}\operatorname{ch}(\kappa|\mathcal{R}|),$$

and using (Eq.5.3)

$$|\dot{Q}(s)| \leq \sqrt{D_{\mathcal{R}}}e^{L_{\mathcal{R}}}\left(1 + \frac{\kappa\,\mathrm{sh}(\kappa|\mathcal{R}|)}{\sqrt{e_{\mathcal{R}}}}\right),\ |\dot{R}(s)| \leq e^{L_{\mathcal{R}}}\left(1 + \frac{\kappa\,\mathrm{sh}(\kappa|\mathcal{R}|)}{\sqrt{e_{\mathcal{R}}}}\right).$$

Finally let T be an arbitrary solution of the original equation. Applying **Remark 5.1** for Q,R and T we find

$$T(s) = T(0)Q(s) + \dot{T}(0)R(s),$$

and hence

$$|T(s)| \leq |T(0)|\sqrt{\frac{D_{\mathcal{R}}}{e_{\mathcal{R}}}}e^{L_{\mathcal{R}}}\,\mathrm{ch}(\kappa|\mathcal{R}|) + |\dot{T}(0)|\frac{1}{\sqrt{e_{\mathcal{R}}}}e^{L_{\mathcal{R}}}\,\mathrm{ch}(\kappa|\mathcal{R}|),$$

as asserted. □

5.3 A result from functional calculus

In this section we will obtain a result using the theory of holomorphic functional calculus of strip type operators. We are grateful to M. Haase for very useful comments on this theory, and refer to his book [26] for all the information necessary in this section.

Let $\mathbb{H}_a = \{z \in \mathbb{C} : |\Im z| < a\}$ denote the symmetric strip of height $a > 0$. If for an (unbounded) operator A on the Banach space $\mathcal{X}$ we have $A \in \mathrm{Strip}(a)$, then we can apply the holomorphic functional calculus of A given by

$$F(A) = \frac{1}{2\pi i}\int_{\gamma_a} dz f(z)\mathfrak{R}(z,A),\ \forall F \in \mathcal{M}[\mathbb{H}_a],$$

where $\gamma_a = \partial\mathbb{H}_a$ oriented positively (counterclockwise), and $\mathfrak{R}(z,A)$ is the resolvent of A for $z \in \mathbb{C}$. Define

$$\mathcal{A}(\mathbb{H}_a) = \{F \in \mathrm{Hol}(\mathbb{H}_a) : \exists N \in \mathbb{N} \text{ s.t. } F = O(|\Re z|^N)\},$$

and

$$\mathcal{A}[\mathbb{H}_a] = \bigcup_{b>a}\mathcal{A}(\mathbb{H}_b).$$

Now let $D_{\Sigma_t} = -\Delta + m^\star(t, \vec{x})$ be the known real lower semi-bounded operator acting on the vector bundle $\mathcal{T}_t$ over a Riemannian manifold Σ_t, and let $K \subset \Sigma_t$ be a compact region. Denote

$$\mathcal{D}(K) = \{f \in \mathcal{D}(\mathcal{T}_t) : \mathrm{supp} f \subset K\}.$$

Then we have the following result.

Proposition 5.4 *For any $F \in \mathcal{A}[\mathbb{H}_0]$ and $f \in \mathcal{D}(K)$ it follows*

$$F(D_{\Sigma_t})f \in \mathcal{D}(K).$$

Proof: Let the nuclear topology be given by $(X_i, 2, l^2)$, i.e., for any $p \in \mathbb{N}_0$ we set

$$(f, h)_p = \sum_{q \le p} (Q_{\alpha,q}(X_i)f, Q_{\alpha,q}(X_i)h)_{L^2}$$

and consider the induced norms $\|.\|_p$. Define the Hilbert spaces

$$\mathcal{H}_p = \overline{\mathcal{D}(K)}^{(,)_p},$$

then by the property of the countably normed spaces we have

$$\mathcal{H}_p \subset \mathcal{H}_q,\ q < p,$$

$$\mathcal{D}(K) = \bigcap_{p=0}^{\infty} \mathcal{H}_p.$$

Fix p, and define the operator D_p on $\mathcal{H}_p$ by setting $D_p f = D_{\Sigma_t} f$ for all $f \in \mathrm{Dom}(D_{\Sigma_t}) \cap \mathcal{H}_p$, then $\mathrm{Dom}(D_p) \supset \mathcal{H}_{p+2}$ is a dense subspace of $\mathcal{H}_p$. Then D_p is a real symmetric operator, and hence by von Neumann's theorem possesses a self-adjoint extension A_p which needs not be lower semi-bounded. The self-adjoint operator A_p has a purely real spectrum, thus $A_p \in \mathrm{Strip}(0)$. Let $\mathcal{A}(\mathbb{H}_a) \ni F(z) = O(|\Re z|^N)$, then for a sufficiently large $a < \lambda \in \mathbb{R}$, the function $e(z) = (z - i\lambda)^{-(N+2)}$ will regularize F on $\mathbb{H}_a$. In particular, we will have $[eF](A_p) \in \mathcal{B}(\mathcal{H}_p)$. Then $F(A_p) = (A_p - i\lambda)^{N+2}[eF](A_p) = [eF](A_p)(A_p - i\lambda)^{N+2}$, from where it follows that $\mathrm{Dom}(A_p^{N+2}) \subset \mathrm{Dom}(F(A_p))$. From the definition of A_p it is clear that $\mathcal{H}_{p+2(N+2)} \subset \mathrm{Dom}(A_p^{N+2})$, whence $\mathcal{H}_{p+2(N+2)} \subset \mathrm{Dom}(F(A_p))$. Thus we have established, that whenever $f \in \mathcal{H}_{p+2(N+2)}$, then necessarily $F(A_p)f \in \mathcal{H}_p$. Now if $f \in \mathcal{D}(K)$, then for any $p \ge 0$ we have $f \in \mathcal{H}_{p+2(N+2)}$, and hence $F(A_p)f \in \mathcal{H}_p$. Meanwhile for any $p \ge 0$, the self-adjoint operator D_{Σ_t} agrees with A_p on $\mathcal{D}(K)$. Therefore also their functional calculi agree, $F(D_{\Sigma_t})f = F(A_p)f \in \mathcal{H}_p$. Thus

$$F(D_{\Sigma_t})f \in \bigcap_{p=0}^{\infty} \mathcal{H}_p = \mathcal{D}(K),$$

which completes the proof. □

Epilogue

Outlook

Several directions of further investigation can be mentioned. For the first chapter it would be interesting to investigate the role of the infrared problems on the observable level. Very recently a proposal has been made in [36] to consider quantum field theory on affine rather than vector bundles which may help to deal with inhomogeneous field equations. It may be interesting to try to generalize the mode decomposition to affine bundles.

The second chapter has many ways to be improved and developed. This concerns mainly complicated problems in harmonic analysis which may be possible to give satisfactory answers when restricted to cosmological situations. Of prime importance would be to obtain Paley-Winer-type results for these spaces.

The chapter three by itself represents an investigation of a particular case. It may be extended in several directions. First, as mentioned earlier, one can try to invoke explicit eigenfunctions of the Laplace operator for vector fields. On the other hand one can extend the harmonic analysis to the LRS spaces and quotient spaces G/Γ.

In chapter 4 one may try to simplify the formulas in special cases and more importantly to obtain a more convenient characterization of the Hadamard property. In particular a necessary and sufficient condition purely in terms of mode coefficients may be found once the Paley-Winer-type result mentioned for the chapter 2 is obtained. And, of course, what remains to do is to construct the states of low energy explicitly and to check whether they are pure, homogeneous and Hadamard. Then one could already try to repeat the calculations of [14] in this generality. Another task is to perform all this analysis also for CAR fields.

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