UNIVERSITY OF NOVA GORICA
SCHOOL OF SCIENCE

PROPERTIES OF NULL HYPERSURFACES

MASTER'S THESIS

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Povzetek

Cilj te magistrske naloge je raziskovanje učinkov, ki jih lahko ima singularna svetlobna hiperploskev na ujemanje časovnih (prostorskih) geodetskih črt. Pri tem želimo razširiti obstoječo teorijo za primer ničelnih geodetskih črt.

V uvodu obravnavamo uporabo singularnih hiperploskev za opis fizikalnih fenomenov, njihovih glavnih razvrstitev, ter opišemo dva obstoječa teoretična pristopa za raziskovanje singularnih ploskev.

V drugem poglavju razvijemo teoretična okvirja in vključimo podroben opis obeh pristopov za primera svetlobnih in časovnih (prostorskih) hiperploskev. V tem poglavju prav tako omenimo možno uporabo obeh teoretskih pristopov za primer, ko hiperploskev vsebuje svetlobni signal s planarno fronto.

V končnem poglavju začnemo z razpravo učinkov, ki jih lahko ima singularna svetlobna hiperploskev na ujemanje časovnih (prostorskih) geodetskih črt. Tu prikažemo nov pristop k izračunu in predstavimo razširitev teorije za primer ujemanja ničelnih geodetskih črt. Na koncu poglavja prikažemo konkretni primer in njegove podobnosti s primeri časovnih geodetskih črt.

Z magistersko nalogo tako predlagamo uporabo novega matematičnega okvirja za opis ujemanja ničelnih geodetskih črt, ki prečkajo singularno ničelno hiperploskev. Rezultate, ki jih pri tem prejmemo, je možno uporabiti pri eksperimentalni fiziki za detekcijo impulznih signalov, ki se nahajajo v singularnih ničelnih hiperploskvah. Kot primer opišemo lastnosti in možnosti detektorja impulznih svetlobnih signalov, ki vključujejo gravitacijske valove.

Ključne besede: singularna hiperploskev, impulzni signal, gravitacijski val, ničelna geodetska črta, časovna (prostorska) geodetska črta
PACS: 04.20.-q, 04.30.-w, 04.30.Db, 04.30.Nk, 04.80.Cc

Abstract

The aim of this thesis is to investigate the effects that a lightlike singular hypersurface can have on a congruence of timelike (spacelike) geodesics and to extend the existing theory to the case of null geodesics.

The introduction discusses the applications of singular hypersurfaces for the description of physical phenomena, their major classfications and includes a short discussion of the two theoretical approaches that exist to study singular hypersurfaces.

The second chapter contains detailed description of these approaches. The theoretical frameworks for both cases of lightlike and timelike (spacelike) hypersurfaces are developed. This chapter also discusses the application of these theories to the case when the hypersurface contains a plane fronted lightlike signal.

The final chapter starts with a discussion of the effects that a lightlike singular hypersurface can have on a congruence of timelike (spacelike) geodesics. A new approach to these calculations is presented together with an extension of the theory to the case of a congruence of null geodesics. At the end of the chapter a concrete example and its similarities with the case of timelike geodesics is discussed.

In conclusion, the thesis suggests a new mathematical framework for describing a congruence of null geodesics crossing a singular null hypersurface. The results may be applied in experimental physics to detect impulsive signals which are located in singular null hypersurfaces and to this end there is a discussion of the properties and possibilities for a detector of impulsive lightlike signals, which include gravitational waves.

Keywords: singular hypersurface, impulsive signal, gravitational wave, null geodesic, timelike (spacelike) geodesic
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Contents

1 Introduction

There are many examples of physical systems in nature where the space-time is described by two metrics joined at a common boundary. Such systems may appear after cataclysmic astrophysical events, such as a supernova or a collision of neutron stars. These systems are used to simulate an exploding white hole, to model an impulsive null signal from a system of neighboring test particles and have many other applications. In general one can choose these two metrics to be either continuous or discontinuous on the boundary hypersurface subject to the condition that the induced metric is unique, but either way the metric's transverse derivative will not be continuous. This always leads to a singularity in the form of a δ-function in the Riemann tensor.

Solutions to the Einstein equations give different results depending on whether the boundary surface is taken to be null or timelike (spacelike). In the first case, when the hypersurface is null, both Weyl and Ricci parts of the Riemann tensor are singular. As it is known, the Weyl part of the Riemann tensor is associated with gravitational waves, whereas the Ricci tensor has non-zero value only in the presence of some material source. Hence, in the case of null boundary hypersurface, both a material source and an impulsive tidal wave can be present. When the boundary surface is timelike, only the Ricci tensor is singular, giving rise only to a matter stress-energy tensor. Depending on the matter distribution, the discussed hypersurfaces are classified into two types: shock waves or boundary surfaces, which arise when there is a jump discontinuity in the density of the stress-energy tensor between the two metrics that they divide and surface layers, otherwise called thin shells, where the density becomes infinite.

There are two different approaches to describe singular hypersurfaces. The first is called the distributional method. In this case a common set of coordinates is used for both sides of the hypersurface. The other method is a generalization of the "cut and paste" approach of Penrose. Here the space-time coordinates on the two sides of the hypersurface can be chosen independently from each other, so in this sense it is a more general approach than the distributional algorithm. It was introduced by Israel to describe timelike hypersurfaces [1] but it was not suitable for the case of null hypersurfaces. In the timelike case, the Israel approach uses the extrinsic curvature of the hypersurface to describe the stress-energy tensor in it. When we move to the null case, the intrinsic metric of the hypersurface space-time becomes degenerate, because the normal vector becomes tangent and there is no distinguishable transverse vector defined. Hence, the extrinsic curvature, which is defined in terms of the metric, is no longer uniquely definable, so it cannot be used to study the hypersurface. This problem was solved and the approach was generalized for the lightlike case by Barrabès and Israel [2] in 1991.

Both of these approaches are thoroughly discussed in the thesis for timelike (spacelike), as well as lightlike hypersurfaces. After having the mathematical description of the hypersurfaces covered, we turn to the question of the effects that an impulsive signal in a singular hypersurface can have on a particle which encounters it. More precisely, we study the geodesic deviation vector between two particles which cross a lightlike hypersurface in a Minkowski space. We distinguish two possibilities corresponding to the cases of timelike and lightlike particles. Although the case of timelike particles has already been studied by Barrabès and Hogan in [5], we suggested a new derivation for the projection of the geodesic deviation vector. Our calculations are more straightforward and lack the technical complexity that exists in [5]. Compared to the timelike case, the case of null particles has not been discussed before. Our calculations are presented in the thesis, where we see that the deviation vector has some interesting features due to the lightlike nature of the particles.

In chapter 2 of the thesis we introduced the general mathematical framework of singular hypersurfaces. It is mostly an overview of the corresponding chapters of [3], with more detailed calculations and explanations. The first two sections of the chapter explain the two algorithms for describing both timelike (spacelike) and lightlike singular hypersurfaces and the third section discusses an example where both of these algorithms are applied. In chapter 3 we have discussed the effects that a lightlike impulsive signal would have on two test particles. Sections 3.1 and 3.2 give an overview of the case where these particles are timelike and also an example of the application of the results [5]. In these sections we have also suggested a new approach for calculating these effects. In the sections 3.3 and 3.4 we give our calculations of the effects that a lightlike impulsive signal will have on a pair of lightlike particles, a case which has not been covered before. Also an example of the application of the results is given, emphasizing the similarities and differences with the timelike case. In Appendix A we give the description of the geodesic deviation vector and very important relations that are being used throughout the thesis. Appendix B is a detailed overview of a derivation of an equation which describes the case of timelike particles [5]. Appendix C gives a proof of an important claim which is often used throughout the chapter 3.

2 General Description of Singular Hypersurfaces

As already mentioned, there are two approaches to describe a hypersurface which is a boundary surface for two domains of different space-time metrics. One of them is the distributional algorithm and the other one is the generalization to the "cut and past" approach of Penrose, introduced by Israel. Israel's approach is easier to apply to the problem, as one can define domains with different metrics in each side of the hypersurface, whereas in the distributional method one has to find a common metric describing both domains, which is not always straightforward. So, in different problems one can use that approach which is more convenient.

Both of the approaches slightly differ in the cases of lightlike and timelike (spacelike) hypersurfaces. The difference comes from the choice of the basis vectors on the hypersurface. In the timelike or spacelike case the normal vector to the surface is not null, but in the lightlike case the normal vector is also tangent to the surface and the metric becomes degenerate. In this case a new vector, the transversal vector has to be defined.

2.1 Distributional Algorithm

In this section we will discuss a space-time manifold $\mathcal{M}$ which is divided into two domains by a hypersurface with a C^0 metric tensor (metric tensor is continuous across the hypersurface but its first derivatives are not). We will denote the domain on the left side of the hypersurface by $\mathcal{M}^+$ and on the right side of the hypersurface by $\mathcal{M}^-$ (see fig. 3.2). Let $\{x^\mu\}$ (μ and other Greek indices take values 0, 1, 2, 3) be the local coordinates on both sides of the hypersurface and $\Phi(x) = 0$ be the equation of the hypersurface, with $\Phi > 0$ corresponding to $\mathcal{M}^+$ and $\Phi < 0$ to $\mathcal{M}^-$. The components of the metric tensor on space-time $\mathcal{M}^+ \bigcup \mathcal{M}^-$ will be denoted by $g^{\pm}_{\mu\nu}$ respectively. Accordingly, every tensor field of any type will be denoted by $+$ or $-$ superscripts on $\mathcal{M}^+$ and $\mathcal{M}^-$. If these tensors differ on both sides of the boundary hypersurface $\mathcal{H}$, the jump $[F] = F^+|_{\mathcal{H}} - F^-|_{\mathcal{H}}$ across $\mathcal{H}$, will not be zero and is an important quantity for our later derivations. In this notation $|_{\mathcal{H}}$ indicates that $F^{\pm}$ should be evaluated on the $\pm$ sides of $\mathcal{H}$.

An important condition on the definition of the metrics should be the first junction condition which states that the metric on the hypersurface should not differ as we approach it from each side. In other words

$$[g_{\mu\nu}] = 0$$

These are general notations for both null and timelike (spacelike) hypersurfaces. We will first discuss the timelike case.

2.1.1 Timelike and Spacelike Hypersurfaces

In this section we will assume that the hypersurface $\mathcal{H}$ is timelike (spacelike) and will denote it by $\mathcal{S}$. So, the jump of some tensor F across $\mathcal{S}$ will be

$$[F] = F^+|_{\mathcal{S}} - F^-|_{\mathcal{S}}. \tag{2.1}$$

We also define a hybrid tensor field $\tilde{F}$ as

$$\tilde{F}(x) = F^+\theta(\Phi) + F^-(1 - \theta(\Phi)), \tag{2.2}$$

where θ is the Heaviside step function which is defined as

$$\theta(\Phi) = \begin{cases} 1 & \Phi > 0 \\ \frac{1}{2} & \Phi = 0 \\ 0 & \Phi < 0 \end{cases} . \tag{2.3}$$

We will assume we are dealing with a continuous metric across $\mathcal{S}$, so in the (2.1) notations $F^\pm = g^\pm$ and we can write

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} \qquad \text{and} \qquad [g_{\mu\nu}] = 0. \tag{2.4}$$

The normal to the hypersurface can be expressed via Φ and is given by

$$n_\mu = \chi^{-1}(x)\partial_\mu\Phi(x) \tag{2.5}$$

where $\chi(x)$ is some normalization factor, adjusted in such a way that ϵ - the normalization factor of n

$$n \cdot n = g^{\mu\nu} n_\mu n_\nu|_\pm = \epsilon \tag{2.6}$$

is $+1(-1)$ when the hypersurface is timelike (spacelike).

Because in this section we are going to construct the Riemann and Einstein tensors for the space-time $\mathcal{M}$, which depend on partial derivatives of $g_{\mu\nu}$, we should also find an equation for a partial derivative of some general F tensor field. That is

$$\begin{aligned} \partial_\mu\tilde{F} &= \theta(\Phi)\partial_\mu F^+ + F^+\partial_\mu\theta(\Phi) + (1-\theta(\Phi))\partial_\mu F^- - F^-\partial_\mu\theta(\Phi) \\ &= \widetilde{\partial_\mu F} + [F]\partial_\mu\theta. \end{aligned} \tag{2.7}$$

For the derivative of the Heaviside step function, we can write

$$\partial_\mu\theta(\Phi) = \partial_\Phi\theta(\Phi)\partial_\mu\Phi = \delta(\Phi)\partial_\mu\Phi = \delta(\Phi)\chi n_\mu,$$

so equation (2.7) becomes

$$\partial_\mu\tilde{F} = \widetilde{\partial_\mu F} + [F]\chi n_\mu\delta(\Phi) \tag{2.8}$$

In addition, we should also note that if we have two F and G tensors defined, then from (2.2) it follows that

$$\tilde{F}\tilde{G} = \widetilde{FG} - [F][G]\,\theta(\Phi)\,\theta(-\Phi).$$

Although we've defined the metric $g_{\mu\nu}$ to be continuous across the hypersurface, its transverse derivative is not. The derivatives in the tangential directions are continuous as the metric is continuous. Thus, the discontinuity of the transverse derivative is directed along the n^α vector and is expressed through a symmetric tensor $\gamma_{\mu\nu}$

$$[\partial_\alpha g_{\mu\nu}] = \epsilon n_\alpha \gamma_{\mu\nu}, \tag{2.9}$$

which is defined only on $\mathcal{S}$ in such a way that its projection on $\mathcal{S}$ is unique. After this requirement is applied, $\gamma_{\mu\nu}$ will be free up to the gauge transformation

$$\gamma_{\mu\nu} \to \gamma'_{\mu\nu} = \gamma_{\mu\nu} + v_\mu n_\nu + n_\mu v_\nu, \tag{2.10}$$

where v is some vector on $\mathcal{S}$. γ is a very important tensor because we will construct Einstein's equation based on the jump of the transversal derivative of the metric, which will lead to the mass-energy tensor of the matter in the hyperspace. The latter will depend on $\gamma_{\mu\nu}$.

To derive the form of Einstein's tensor we need to start from the Christoffel symbols. Using (2.4) and (2.8) we can obtain

$$\partial_\rho \tilde{g}_{\mu\nu} = \tilde{\partial_\rho g_{\mu\nu}}, \tag{2.11}$$

from which it follows that

$$\begin{aligned} \Gamma^\lambda_{\mu\nu} &= \frac{1}{2} g^{\lambda\rho}[\partial_\mu \tilde{g}_{\rho\nu} + \partial_\nu \tilde{g}_{\rho\mu} - \partial_\rho \tilde{g}_{\mu\nu}] \\ &= \frac{1}{2} g^{\lambda\rho}[\tilde{\partial_\mu g_{\rho\nu}} + \tilde{\partial_\nu g_{\rho\mu}} - \tilde{\partial_\rho g_{\mu\nu}}] \\ &= \tilde{\Gamma}^\lambda_{\mu\nu} \end{aligned} \tag{2.12}$$

For $[\Gamma^\lambda_{\mu\nu}]$ we will get

$$\begin{aligned} \left[\Gamma^\lambda_{\mu\nu}\right] &= \frac{1}{2} g^{\lambda\rho}([\partial_\mu g_{\rho\nu}] + [\partial_\nu g_{\rho\mu}] - [\partial_\rho g_{\mu\nu}]) \\ &= \frac{1}{2} g^{\lambda\rho}(\epsilon n_\mu \gamma_{\rho\nu} + \epsilon n_\nu \gamma_{\rho\mu} - \epsilon n_\rho \gamma_{\mu\nu}) \\ &= \frac{1}{2}\epsilon(n_\mu \gamma^\lambda_\nu + n_\nu \gamma^\gamma_\mu - n^\lambda \gamma_\mu \nu) = \\ &= \epsilon\left(\gamma^\lambda_{(\mu} n_{\nu)} - \frac{1}{2}\gamma_{\mu\nu} n^\lambda\right) \end{aligned} \tag{2.13}$$

The brackets around indices denote symmetrization over those indices. Summarizing last two results we can write

$$\Gamma^\lambda_{\mu\nu} = \tilde{\Gamma}^\lambda_{\mu\nu} \qquad \text{and} \qquad \left[\Gamma^\lambda_{\mu\nu}\right] = \epsilon\left(\gamma^\lambda_{(\mu} n_{\nu)} - \frac{1}{2}\gamma_{\mu\nu} n^\lambda\right). \tag{2.14}$$

Now we can calculate the Riemann tensor using (2.14). The Riemann tensor is expressed through Christoffel symbols according to the following equation.

$$R_{k\lambda\mu\nu} = \partial_\mu \tilde{\Gamma}_{k\lambda\nu} - \partial_\nu \tilde{\Gamma}_{k\lambda\mu} + \tilde{\Gamma}_{k\mu\rho}\tilde{\Gamma}^\rho_{\nu\lambda} - \tilde{\Gamma}_{k\nu\rho}\tilde{\Gamma}^\rho_{\mu\lambda} \tag{2.15}$$

The tilded Riemann tensor will be

$$\tilde{R}_{k\lambda\mu\nu} = \partial_\mu \tilde{\Gamma}_{k\lambda\nu} - \partial_\nu \tilde{\Gamma}_{k\lambda\mu} + \tilde{\Gamma}_{k\mu\rho}\tilde{\Gamma}^\rho_{\ \nu\lambda} - \tilde{\Gamma}_{k\nu\rho}\tilde{\Gamma}^\rho_{\ \rho\mu\lambda} \tag{2.16}$$

From (2.8) and (2.14) we can deduce that

$$\partial_\mu \tilde{\Gamma}_{k\lambda\nu} = \partial_\mu \tilde{\Gamma}_{k\lambda\nu} - [\Gamma_{k\lambda\nu}]\chi n_\mu \delta(\Phi).$$

So, plugging this into the tilded Riemann equation we find

$$\begin{aligned}\tilde{R}_{k\lambda\mu\nu} &= \partial_\mu \tilde{\Gamma}_{k\lambda\nu} - \partial_\nu \tilde{\Gamma}_{k\lambda\mu} + \tilde{\Gamma}_{k\mu\rho}\tilde{\Gamma}_{\rho\nu\lambda} - \tilde{\Gamma}_{k\nu\rho}\tilde{\Gamma}_{\rho\mu\lambda} \\ &\quad - [\Gamma_{k\lambda\nu}]\chi n_\mu \delta(\Phi) + [\Gamma_{k\lambda\mu}]\chi n_\nu \delta(\Phi) \\ &= R_{k\lambda\mu\nu} - [\Gamma_{k\lambda\nu}]\chi n_\mu \delta(\Phi) + [\Gamma_{k\lambda\mu}]\chi n_\nu \delta(\Phi),\end{aligned} \tag{2.17}$$

where terms that vanish distributionally have been ignored. Taking into account the following equality, where square brackets around indices denote skew-symmetrization,

$$\begin{aligned}\hat{R}_{k\lambda\mu\nu} \equiv 2n_{[k}\gamma_{\lambda][\mu}n_{\nu]} &= n_k\gamma_{\lambda[\mu}n_{\nu]} - n_\lambda\gamma_{k[\mu}n_{\nu]} \\ &= \frac{1}{2}(n_k\gamma_{\lambda\mu}n_\nu - n_k\gamma_{\lambda\nu}n_\mu - n_\lambda\gamma_{k\mu}n_\nu + n_\lambda\gamma_{k\nu}n_\mu),\end{aligned}$$

the last two terms in tilde-Riemann equation can be simplified.

$$\begin{aligned}&[\Gamma_{k\lambda\mu}]\,\chi n_\nu \delta(\Phi) - [\Gamma_{k\lambda\nu}]\chi n_\mu \delta(\Phi) \\ &\quad = \epsilon\chi\delta(\Phi)[\gamma_{k(\lambda}n_{\mu)}n_\nu - \frac{1}{2}\gamma_{\lambda\mu}n_k n_\nu - \gamma_{k(\lambda}n_{\nu)}n_\mu + \frac{1}{2}\gamma_{\lambda\nu}n_k n_\mu] \\ &\quad = \epsilon\chi\delta(\Phi)\frac{1}{2}[\gamma_{k\lambda}n_\mu n_\nu - \gamma_{k\lambda}n_\nu n_\mu - n_k\gamma_{\lambda\mu}n_\nu \\ &\qquad\qquad + n_k\gamma_{\lambda\nu}n_\mu + n_\lambda\gamma_{k\mu}n_\nu - n_\lambda\gamma_{k\nu}n_\mu] \\ &\quad = -\epsilon\chi\delta(\Phi)2n_{[k}\gamma_{\lambda][\mu}n_{\nu]} = -\epsilon\chi\delta(\Phi)\hat{R}_{k\lambda\mu\nu}.\end{aligned}$$

Plugging this result into (2.17) will give

$$R_{k\lambda\mu\nu} = \tilde{R}_{k\lambda\mu\nu} + \hat{R}_{k\lambda\mu\nu}\epsilon\chi\delta(\Phi) \tag{2.18}$$

Similar decompositions into a tilde-term and a Dirac δ-function term can be derived for the Ricci tensor $R_{\mu\nu}$ and the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$

$$R_{\mu\nu} = \tilde{R}_{\mu\nu} + \hat{R}_{\mu\nu}\epsilon\chi\delta(\Phi), \tag{2.19}$$

$$G_{\mu\nu} = \tilde{G}_{\mu\nu} + \hat{G}_{\mu\nu}\epsilon\chi\delta(\Phi). \tag{2.20}$$

The singular part of the Einstein tensor is given by

$$\hat{G}_{\mu\nu} = \gamma_{(\mu}n_{\nu)} - \frac{\gamma}{2}n_\mu n_\nu - \frac{\gamma^\dagger}{2}g_{\mu\nu} - \frac{\epsilon}{2}(\gamma_{\mu\nu} - \gamma g_{\mu\nu}), \tag{2.21}$$

where

$$\gamma \equiv g^{\mu\nu}\gamma_{\mu\nu}, \qquad \gamma_\mu \equiv \gamma_{\mu\nu}n^\nu, \qquad \gamma^\dagger \equiv \gamma_{\mu\nu}n^\mu n^\nu = \gamma_\mu n^\mu \tag{2.22}$$

Singularities in the Riemann and Einstein tensors in the form of a Dirac δ-function do not imply any infinities in the motion equation of a particle crossing the hypersurface as we will show in later sections.

Finally, after deriving Einstein tensor, we can find the stress-energy tensor using Einstein equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{2.23}$$

where Λ is the cosmological constant. We can see from here that the stress energy tensor can also be decomposed into a tilde and a singular term.

$$T_{\mu\nu} = \tilde{T}_{\mu\nu} + S_{\mu\nu}\chi\delta(\Phi). \tag{2.24}$$

Plugging this into (2.23) will give

$$8\pi S_{\mu\nu} = \epsilon\hat{G}_{\mu\nu}. \tag{2.25}$$

The singular term in (2.24) indicates the presence of a thin shell and $S_{\mu\nu}$ is intrinsic to $\mathcal{S}$ as it satisfies the $S_{\mu\nu}n^{\nu} = 0$ condition.

2.1.2 Lightlike Hypersurfaces

The lightlike hypersurface case is very similar to the timelike case. Here we will denote the null hypersurface on the boundary of two $\mathcal{M}^{\pm}$ domains by $\mathcal{N}$. We also define a hybrid tensor $\tilde{F}$ as follows

$$\tilde{F}(x) = F^{+}\theta(\Phi) + F^{-}(1 - \theta(\Phi)), \tag{2.26}$$

where $\theta(\Phi)$ is again Heaviside step function. The metric tensor will satisfy the (2.4) conditions as before, which state

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} \qquad \text{and} \qquad [g_{\mu\nu}] = 0. \tag{2.27}$$

Following the same considerations, we can derive the same equation (2.8) for the null hypersurface case

$$\partial_{\mu}\tilde{F} = \widetilde{\partial_{\mu}F} + [F]\chi n_{\mu}\delta(\Phi), \tag{2.28}$$

where χ is some arbitrary function. The difference between lightlike and timelike cases comes, of course, from the fact that the normal vector to the surface in the first case is null. The normal vector is

$$n^{\mu} = \chi^{-1}(x)g^{\mu\nu}\partial_{\nu}\Phi(x) \tag{2.29}$$

and the null condition is

$$n \cdot n \equiv g_{\mu\nu}n^{\mu}n^{\nu}|_{\pm} = 0. \tag{2.30}$$

As it has already been explained, this null vector can not be used to describe the extrinsic properties of $\mathcal{N}$ as embedded in $\mathcal{M}$. For this reason a transversal vector N is introduced on $\mathcal{N}$, which points out of $\mathcal{N}$

$$N \cdot n \equiv \eta^{-1} \neq 0, \qquad N^{+}_{\mu} = N^{-}_{\nu} \equiv N_{\mu}. \tag{2.31}$$

N is not uniquely defined by (2.31). The scalar product (2.31) is invariant under the gauge transformation

$$N \to N' = N + v, \tag{2.32}$$

where v is an arbitrary vector tangent to the hypersurface $\mathcal{N}$.

As in the case of timelike hypersurfaces, here we also describe the discontinuities in the transverse derivatives of the metric tensor by a symmetric tensor $\gamma_{\mu\nu}$

$$[\partial_\alpha g_{\mu\nu}] = \eta n_\alpha \gamma_{\mu\nu}. \tag{2.33}$$

Multiplying both sides with N^α will give

$$N^\alpha [\partial_\alpha g_{\mu\nu}] = \gamma_{\mu\nu}, \tag{2.34}$$

but it is free up to the gauge transformation

$$\gamma_{\mu\nu} \to \gamma'_{\mu\nu} = \gamma_{\mu\nu} + v_\mu n_\nu + n_\mu v_\nu, \tag{2.35}$$

where v is a four-dimensional vector field defined on $\mathcal{N}$.

After going trough a similar procedure as in (2.12) and (2.13), we can obtain equations equivalent to (2.14).

$$\Gamma^\lambda_{\mu\nu} = \tilde{\Gamma}^\lambda_{\mu\nu} \qquad \text{and} \qquad \left[\Gamma^\lambda_{\mu\nu}\right] = \eta \left(\gamma^\lambda_{(\mu} n_{\nu)} - \frac{1}{2} \gamma_{\mu\nu} n^\lambda \right). \tag{2.36}$$

Similarly we find

$$R_{k\lambda\mu\nu} = \tilde{R}_{k\lambda\mu\nu} + \hat{R}_{k\lambda\mu\nu} \epsilon \chi \delta(\Phi), \tag{2.37}$$

$$R_{\mu\nu} = \tilde{R}_{\mu\nu} + \hat{R}_{\mu\nu} \epsilon \chi \delta(\Phi), \tag{2.38}$$

$$G_{\mu\nu} = \tilde{G}_{\mu\nu} + \hat{G}_{\mu\nu} \epsilon \chi \delta(\Phi). \tag{2.39}$$

with the following hat-coefficients

$$\hat{R}_{k\lambda\mu\nu} = 2 n_{[k} \gamma_{\lambda][\mu} n_{\nu]}, \tag{2.40}$$

$$\hat{R}_{\mu\nu} = \gamma_{(\mu} n_{\nu)} - \frac{\gamma}{2} n_\mu n_\nu, \tag{2.41}$$

$$\hat{G}_{\mu\nu} = \gamma_{(\mu} n_{\nu)} - \frac{\gamma}{2} n_\mu n_\nu - \frac{\gamma^\dagger}{2} g_{\mu\nu}, \tag{2.42}$$

where

$$\gamma \equiv g^{\mu\nu} \gamma_{\mu\nu}, \qquad \gamma_\mu \equiv \gamma_{\mu\nu} n^\nu, \qquad \gamma^\dagger \equiv \gamma_{\mu\nu} n^\mu n^\nu = \gamma_\mu n^\mu \tag{2.43}$$

From the form of the Einstein tensor we conclude that the stress-energy tensor will contain two terms one of which proportional to the Dirac δ-function.

$$T_{\mu\nu} = \tilde{T}_{\mu\nu} + S_{\mu\nu} \eta \chi \delta(\Phi). \tag{2.44}$$

The tilde-term of the stress-energy tensor corresponds to the matter content $T^\pm_{\mu\nu}$ of the exterior domains $\mathcal{M}^\pm$. The second term corresponds to the matter in the singular hypersurface, which is actually a shell of lightlike matter. The stress-energy tensor on the null shell is

$$T_{\mu\nu}|_{\mathcal{N}} = S_{\mu\nu}\eta\chi\delta(\Phi)$$

where $S_{\mu\nu}$ is given by

$$16\pi S_{\mu\nu} = -\gamma n_\mu n_\nu - \gamma^\dagger g_{\mu\nu} + 2\gamma_{(\mu} n_{\nu)} = 2\hat{G}_{\mu\nu}. \tag{2.45}$$

The three terms in the stress-energy tensor in the last equation, if taken on the hypersurface $\mathcal{N}$, can be interpreted as being related to the energy density, the isotropic tensor and the energy current respectively, as showed in the next section.

2.2 Extrinsic Curvature Algorithm

The second approach for constructing two domains with a common boundary surface involves specifying two different local coordinate systems $\{x^\mu_\pm\}$ in two domains $\mathcal{M}^\pm$ of the space respectively and introducing intrinsic coordinates $\{\xi^a\}$ (a takes values 1, 2, 3)on their common hypersurface $\mathcal{H}$ (figure 3.2).

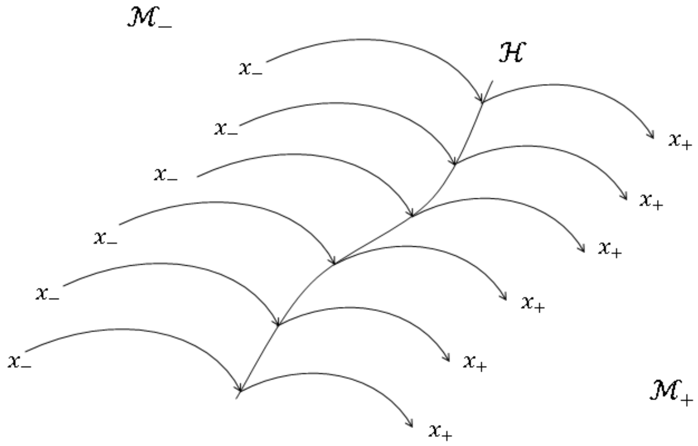

Figure 2.1: Diagram of a hypersurface $\mathcal{H}$, which divides the space-time into two domains $\mathcal{M}^\pm$, each with its own coordinate system $\{x^\mu_\pm\}$.

2.2.1 Timelike and Spacelike Hypersurfaces

In the timelike (spacelike) case, we will denote the hypersurface by $\mathcal{S}$. The tangent vectors $e_{(a)}$ to $\mathcal{S}$ have components

$$e^\mu_{(a)}|_\pm = \frac{\partial x^\mu_\pm}{\partial \xi^a}. \tag{2.46}$$

The completeness relation with the basis $\{n, e_{(a)}\}$ on the hypersurface is

$$g^{\alpha\beta} = g^{ab} e^\alpha_{(a)} e^\beta_{(b)} + \epsilon n^\alpha n^\beta, \tag{2.47}$$

where g^{ab} is the inverse of the induced metric

$$g_{ab} \equiv e_{(a)} \cdot e_{(b)}|_{\pm} = g_{\mu\nu} e^{\mu}_{(a)} e^{\nu}_{(b)}|_{\pm}. \tag{2.48}$$

The extrinsic curvature is defined by

$$K_{ab} = -n_{\mu} e^{\mu}_{(a)|\lambda} e^{\lambda}_{(b)}, \tag{2.49}$$

where the stroke denotes covariant differentiation with respect to the Riemannian connection calculated with the metric tensor on either side of $\mathcal{S}$. Extrinsic curvature has the following relation

$$\gamma_{ab} \equiv 2[K_{ab}] \tag{2.50}$$

with γ_{ab}, which is the projection of the tensor $\gamma_{\mu\nu}$ on $\mathcal{S}$, defined in subsection 2.1.1.

$$\gamma_{ab} = \gamma_{\mu\nu} e^{\mu}_{(a)} e^{\nu}_{(b)}. \tag{2.51}$$

Similarly, the projection of the stress-energy tensor defined in the same section is given by $S_{ab} = S_{\mu\nu} e^{\mu}_{(a)} e^{\nu}_{(b)}$ and the projection of the singular parts of $G_{\mu\nu}$ are given by $\hat{G}_{ab} = \hat{G}_{\mu\nu} e^{\mu}_{(a)} e^{\nu}_{(b)}$. So using equation (2.25) we can write

$$8\pi S_{ab} = \epsilon \hat{G_{ab}}. \tag{2.52}$$

Using Gauss-Codazzi equations and their contracted form together with (2.21) and working in the gauge where $\gamma_{\mu} = 0$ we find

$$8\pi S_{ab} = -[K_{ab}] + [K] g_{ab}. \tag{2.53}$$

This equation describes the evolution of the shell after the stress-energy tensor has been fixed.

2.2.2 Lightlike Hypersurfaces

Three tangent vectors $e_{(a)} = \partial/\partial\xi^a$ to the null hypersurface $\mathcal{N}$ make the basis on $\mathcal{N}$ and have components

$$e^{\mu}_{(a)}|_{\pm} = \frac{\partial x^{\mu}_{\pm}}{\partial \xi^a}. \tag{2.54}$$

As the two domains $\mathcal{M}^{\pm}$ now have possibly different metrics and distinct coordinate systems, we must impose the condition, that there is a unique metric induced on the hypersurface $\mathcal{N}$:

$$g_{ab} \equiv e_{(a)} \cdot e_{(b)} = g_{\mu\nu} e^{\mu}_{(a)} e^{\nu}_{(b)}|_{\pm} \tag{2.55}$$

This metric is degenerate and has rank 2 as the hypersurface it describes ($\mathcal{N}$) is null. The normal vector to $\mathcal{N}$ is defined by (2.30) and is normal to the basis vectors (2.54)

$$n \cdot e_{(a)}|_{\pm} = 0 \tag{2.56}$$

In general for a null normal vector we can write

$$n^\mu = n^a e^\mu_{(a)} \tag{2.57}$$

and from (2.56)

$$n \cdot e_{(a)}|_\pm = 0 = g_{\mu\nu} n^\mu e^\nu_{(a)} = g_{\mu\nu} e^\nu_{(a)} e^\mu_{(b)} n^b = g_{ab} n^b, \tag{2.58}$$

so

$$g_{ab} n^b = 0. \tag{2.59}$$

Because of the degeneracy of the metric on $\mathcal{N}$, we have to define a new transverse vector similar to the one we've defined when we were discussing the distributional algorithm. In this case we can define it using the coordinate system on the hypersurface.

$$N \cdot n|_\pm = n^a N_a = \eta^{-1} \neq 0, \qquad N \cdot e_{(a)}|_+ = N \cdot e_{(a)}|_- \equiv N_a \tag{2.60}$$

From here we will proceed in a similar way we did in the subsection 2.2.1. There we've used the extrinsic curvature tensor to define the projection of the $\gamma_{\mu\nu}$ tensor on the hypersurface. Because here we are discussing a null hypersurface, the normal is also tangential, and K_{ab} extrinsic curvature, defined by (2.49), retains only tangential derivatives. But, tangential derivatives are continuous, so $[K_{ab}]$ vanishes identically. For this reason we define transverse curvature

$$\mathcal{K}^\pm_{ab} = -N_\mu e^\mu_{(a)|\nu} e^\nu_{(b)}|_\pm \tag{2.61}$$

A transverse curvature can also be defined in the case of a timelike hypersurface. Using $\mathcal{K}^\pm_{ab}$ we can define a symmetric tensor

$$\gamma_{ab} \equiv 2[\mathcal{K}_{ab}] \tag{2.62}$$

Although $\mathcal{K}_{ab}$ depends on the choice of N_μ, the jump of $\mathcal{K}_{ab}$ does not, because under the transformation $N \to N' = N + v^a e_{(a)}$, where $v^a \in \mathcal{N}$, it transforms according to

$$\mathcal{K}_{ab} \to \mathcal{K}'_{ab} = \mathcal{K}_{ab} - v^c \Gamma_{cab}, \tag{2.63}$$

where $\Gamma_{cab} = e^\mu_{(c)} e_{(a)\mu|\nu} e^\nu_{(ab)}$ are Ricci rotation coefficients. As we can see the latter only contains tangential derivatives of the metric, so it is continuous across $\mathcal{N}$ and the jump $[\mathcal{K}_{ab}]$ is invariant. Although $\mathcal{K}_{ab}$ is not a tensor, the jump $[\mathcal{K}_{ab}]$ is a tensor.

There exists a relation between the symmetric tensor $\gamma_{\mu\nu}$ defined in the subsection 2.1.2 and γ_{ab}

$$\gamma_{\mu\nu} e^\mu_{(a)} e^\nu_{(b)} = \gamma_{ab}. \tag{2.64}$$

Now we can discuss the contravariant components surface stress-energy tensor for which we can write

$$S^{\mu\nu} = S^{ab} e^\mu_{(a)} e^\nu_{(b)}. \tag{2.65}$$

Because g_{ab} is degenerate, it can not be inverted to raise the Latin letters. But using the fact that the four vectors $(N^\mu, e^\mu_{(a)})$ form a basis we can write the completeness relation

$$g^{\mu\nu} = g^{ab}_* e^\mu_{(a)} e^\nu_{(b)} + 2\eta n^a e^{(\mu}_{(a)} N^{\nu)} \tag{2.66}$$

where g_*^{ab} will be defined up to the following conditions

$$g_*^{ac} g_{cb} = \delta_b^a - \eta n^a N_b \tag{2.67}$$

$$g_*^{ab} N_b + \eta n^a (N \cdot N) = 0 \tag{2.68}$$

Using the stress-energy tensor (2.45) obtained through the distributional algorithm for the null hypersurface case and also introducing (2.43) and (2.66) we can obtain

$$16\pi S^{ab} = -(\gamma_{cd} g_*^{cd}) n^a n^b - (\gamma_{cd} n^c n^d) g_*^{ab} + (g_*^{ac} n^b n^d + g_*^{bc} n^a n^d)\gamma_{cd}. \tag{2.69}$$

Depending on the choice of the transversal vector (2.32), g_*^{ab} has the freedom

$$g_*^{ab} \to g_*^{ab} - \eta(v^a n^b + v^b n^a), \tag{2.70}$$

but the contravariant components S^{ab} of the surface stress-energy tensor are independent of this choice. Thus S^{ab} are intrinsic components.

Introducing the following relations

$$\gamma^\dagger = \gamma_{\mu\nu} n^\mu n^\nu = \gamma_{ab} n^a n^b, \tag{2.71}$$

$$\gamma_a = \gamma_{ab} n^b = \gamma_\mu e^\mu_{(a)}, \tag{2.72}$$

$$\gamma = \gamma_{\mu\nu} g^{\mu\nu} = \gamma_{ab} g_*^{ab} + 2\eta N^\mu \gamma_\mu \tag{2.73}$$

(2.69) can be modified to take the following form

$$S^{ab} = \mu n^a n^b + P g_*^{ab} + Q^a n^b + Q^b n^a, \tag{2.74}$$

where

$$\mu = -\frac{1}{16\pi}\gamma_{cd} g_*^{cd}, \qquad P = -\frac{1}{16\pi}\gamma^\dagger, \qquad Q^a = \frac{1}{16\pi} g_*^{ac}\gamma_c \tag{2.75}$$

These quantities can be associated with surface energy density, surface isotropic pressure and surface energy current respectively [4].

2.3 A Plane Fronted Lightlike Signal

A particularly rich example of matching two metrics along a null hypersurface is that of a plane fronted impulsive lightlike signal, which combines an impulsive gravitational wave and a lightlike shell [3]. We will assume that the space in which the signal propagates is a flat space-time and the history $\mathcal{N}$ of the signal is a null hypersurface. Choosing such a configuration, we will apply the distributional algorithm and Israel approach to describe the stress-energy tensor on the hypersurface. So, the two $\mathcal{M}^+$ and $\mathcal{M}^-$ domains of the space described by $u_+ \geq 0$ and $u_- \leq 0$ have the following line elements

$$\mathrm{d}s_\pm^2 = -2\,\mathrm{d}u_\pm\,\mathrm{d}v_\pm + \mathrm{d}x_\pm^2 + \mathrm{d}y_\pm^2. \tag{2.76}$$

The coordinate system is chosen to be $x_\pm^\mu = (u_\pm, v_\pm, x_\pm, y_\pm)$ with $\mu = 0, 1, 2, 3$. The equation of the null hypersurface $\mathcal{N}$ is $u_+ = u_- = 0$. The matching conditions of $\mathcal{M}^+$ and $\mathcal{M}^-$ on $\mathcal{N}$ will be defined as follows

$$x_+ = x_-, \qquad y_+ = y_-, \qquad v_+ = F(v_-, x_-, y_-), \tag{2.77}$$

where F is an arbitrary function, with the only restriction being $\partial F/\partial v_- \neq 0$. Applying (2.77) on the induced line element on $\mathcal{N}$ will give $\mathrm{d}s^2|_{\mathcal{N}} = \mathrm{d}x_+^2 + \mathrm{d}y_+^2 = \mathrm{d}x_-^2 + \mathrm{d}y_-^2$.

We will first discuss the application of the distributional algorithm but to proceed further we need to define a common coordinate system $x^\mu = (u, v, x, y)$ and a metric $\mathrm{d}s^2$ for the whole space-time instead of $x_\pm^\mu$ and $\mathrm{d}s_\pm^2$. We are actually free to take x_-^μ as such a coordinate system

$$u_- = u, \qquad v_- = v, \qquad x_- = x, \qquad y_- = y, \tag{2.78}$$

and express x_+^μ through it. We will assume that the coordinate transformation equations from x_-^μ to x_+^μ are linear in u_-^μ, for a reason discussed later:

$$\begin{aligned} x_+ &= A_1 + A_2\, u_-, \\ y_+ &= B_1 + B_2\, u_-, \\ u_+ &= C_1 + C_2\, u_-, \\ v_+ &= D_1 + D_2\, u_-, \end{aligned} \tag{2.79}$$

where, in general, all the coefficients of x_-^μ depend on v_-, x_-, y_-. Applying matching conditions (2.77) will remove most of the terms in the latter transformation equations, leaving us with four unknown constants

$$\begin{aligned} x_+ &= x_- + A\, u_-, \\ y_+ &= y_- + B\, u_-, \\ u_+ &= C u_-, \\ v_+ &= F + D\, u_-. \end{aligned} \tag{2.80}$$

To find these constants we need to apply the condition that the line elements of the two space-time domains are equal on the null hypersurface:

$$\mathrm{d}s_+^2|_{u=0} = \mathrm{d}s_-^2|_{u=0}. \tag{2.81}$$

To construct $\mathrm{d}s_+^2|_{u=0}$ from (2.86) we need find the first and second order differentials to these equations and take $u = 0$, which will yield

$$\begin{aligned} \mathrm{d}x_+^2 &= \mathrm{d}x_-^2 + A^2\, \mathrm{d}u_-^2 + 2A\, \mathrm{d}x_-\, \mathrm{d}u_-, \\ \mathrm{d}y_+^2 &= \mathrm{d}y_-^2 + B^2\, \mathrm{d}u_-^2 + 2B\, \mathrm{d}y_-\, \mathrm{d}u_-, \\ \mathrm{d}u_+\, \mathrm{d}v_+ &= C\, \mathrm{d}u_-\, \mathrm{d}F + CD\, \mathrm{d}u_-^2. \end{aligned} \tag{2.82}$$

and from these equations we find

$$\mathrm{d}s_+^2|_{u=0} = \mathrm{d}x_-^2 + \mathrm{d}y_-^2 + \mathrm{d}\Omega_-\, \mathrm{d}u_- = \mathrm{d}s_-^2|_{u=0}, \tag{2.83}$$

where

$$\mathrm{d}\Omega_- = (A^2\, \mathrm{d}u_- + 2A\, \mathrm{d}x_- + B^2\, \mathrm{d}u_- + 2B\, \mathrm{d}y_- - 2CD\, \mathrm{d}u_- - 2C\, \mathrm{d}F).$$

So, from (2.81) follows that $\mathrm{d}\Omega_- = -2\, \mathrm{d}v_-$:

$$A^2\, \mathrm{d}u_- + 2A\, \mathrm{d}x_- + B^2\, \mathrm{d}u_- + 2B\, \mathrm{d}y_- - 2CD\, \mathrm{d}u_- - 2C\, \mathrm{d}F = -2\, \mathrm{d}v_- \tag{2.84}$$

Dividing this equation by $\mathrm{d}u_-, \mathrm{d}v_-, \mathrm{d}x_-$ and $\mathrm{d}y_-$ one by one, we will find the following relations for the constants A, B, C, D.

$$\begin{aligned} A &= CF_x, & B &= CF_y, \\ C &= \frac{1}{F_v}, & D &= \frac{A^2 + B^2}{2C}. \end{aligned} \tag{2.85}$$

Here the variables on subscript of F denote partial derivatives on that coordinate. Finally, by plugging these values back into (2.86) we will find

$$\begin{aligned} x_+ &= x + u\frac{F_x}{F_v}, \\ y_+ &= y + u\frac{F_y}{F_v}, \\ u_+ &= \frac{u}{F_v}, \\ v_+ &= F + \frac{u}{2F_v}(F_x^2 + F_y^2). \end{aligned} \tag{2.86}$$

Taking this into account we can express the continuous metric on the space-time $\mathcal{M}^+ \bigcup \mathcal{M}^-$ via the line element [5] (using the fact that $g_{\mu\nu} = \tilde{g_{\mu\nu}} = g^+_{\mu\nu}\theta(u) + g^-_{\mu\nu}[1 - \theta(u)]$)

$$\begin{aligned} \mathrm{d}s^2 = -2\,\mathrm{d}v\left(\mathrm{d}u - \frac{u\theta(u)}{F_v}\,\mathrm{d}F_v\right) \\ + \left(\mathrm{d}x + \frac{u\theta(u)}{F_v}\,\mathrm{d}F_x\right)^2 + \left(\mathrm{d}y + \frac{u\theta(u)}{F_v}\,\mathrm{d}F_y\right)^2, \end{aligned} \tag{2.87}$$

where $\theta(u)$ is the Heaviside step function.

2.3.1 Distributional Algorithm

To discuss the application of the distributional algorithm on the space-time described above, we will work in the local coordinates (u, v, x, y) with the metric tensor $g_{\mu\nu}$ of the line element (2.87). We take as the normal vector to $\mathcal{N}$ to be $n_\mu = -u_{,\mu}$, where a comma denotes a partial differentiation with respect to the local coordinates, as in the rest of the thesis. We will assume that the transversal N^μ satisfies $N^\mu n_\mu = -1$ condition. Accordingly, the contravariant components of the normal to $\mathcal{N}$ will be $n^\mu = \delta^\mu_1$ and a convenient choice for the transversal is $N^\mu = \delta^\mu_0$.

After having everything defined we can continue by finding $\gamma_{\mu\nu}$ defined on $\mathcal{N}$. As only the u component of the transversal vector is not 0, using (2.34) we find

$$\gamma_{\mu\nu} = \left[\frac{\partial g_{\mu\nu}}{\partial u}\right] = \left.\frac{\partial g_{\mu\nu}}{\partial u}\right|_+ - \left.\frac{\partial g_{\mu\nu}}{\partial u}\right|_-. \tag{2.88}$$

Now we can see why we had chosen x^μ_+ to be linear on u. $\gamma_{\mu\nu}$ is, of course, defined on $\mathcal{N}$, where $u = 0$. If there were higher order terms on u in the transformation equations (2.79), they wouldn't give rise to any new terms in the equation (2.82), as these terms would all be proportional to u. Thus, the first term on the right hand side of the equation (2.88) also wouldn't change if there were higher order terms on u in the transformation equations (2.79). And as the stress-energy tensor on the hypersurface is defined through $\gamma_{\mu\nu}$, it also wouldn't change and so the higher terms in u would have no physical contribution on the shell.

Using (2.88) we can calculate

$$\gamma_\mu = \gamma_{\mu 2} = \left(0, \frac{2F_{vv}}{F_v}, \frac{2F_{xv}}{F_v}, \frac{2F_{yv}}{F_v}\right) \tag{2.89}$$

$$\gamma = \frac{2}{F_v}(F_{xx} + F_{yy}), \qquad \gamma^\dagger = \gamma_{22} = \frac{2F_{vv}}{F_v} \tag{2.90}$$

The surface stress-energy tensor can be obtained from (2.45).

2.3.2 Extrinsic Curvature Algorithm

In the case of using the extrinsic curvature algorithm to study the plane fronted impulsive lightlike signal, we don't need a common metric for both sides of the null hypersurface $\mathcal{N}$, like (2.87). We only need the line elements (2.76) on the $\mathcal{M}^\pm$ respectively, the local coordinates $\{x^\mu_\pm\}$ and the matching conditions (2.77). We chose the intrinsic coordinates on $\mathcal{N}$ to be $\xi^a = (v_-, x_-, y_-) = (v, x, y)$, using (2.78). The tangent basis vectors $e_{(a)} = \partial/\partial\xi$ can be calculated using (2.77) and (2.78). They can be calculated to be $e^\mu_{(a)}|_- = \delta^\mu_a$ and $e^\mu_{(1)}|_+ = (0, F_v, 0, 0)$, $e^\mu_{(2)}|_+ = (0, F_x, 1, 0)$, $e^\mu_{(3)}|_+ = (0, F_y, 0, 1)$. The normal is taken to be $n^\mu = e^\mu_{(1)}$. The transversal vector can be uniquely defined by $N^\mu n_\mu = -1$, $N^\mu N_\mu = 0$ and $N_\mu e^\mu_{(A)} = 0$, with $A = 2, 3$ and $\xi^A = (x, y)$. After calculations we can find that $N^\mu|_- = \delta^\mu_0$ and $N^\mu|_+ = (F_v^{-1}, 2F_v^{-1}(F_x^2 + F_y^2), F_v^{-1}F_x, F_v^{-1}F_y)$.

Using the definitions above, we can calculate the transverse curvatures $\mathcal{K}^\pm_{ab}$ from (2.61),

$$\gamma_{ab} = \frac{2F_{ab}}{F_v} \tag{2.91}$$

where the subscripts denote partial derivatives on F. We can choose the pseudo-inverse metric to have components $g^{1b}_* = 0$, $g^{AB}_* = \delta_{AB}$, because we have $g_{1b} = 0$, $g_{AB} = \delta_{AB}$. Using this and the derived equation (2.74) for the stress-energy tensor we can calculate it with

$$\mu = -\frac{1}{8\pi F_v}(F_{xx} + F_{yy}), \qquad P = -\frac{F_{vv}}{8\pi F_v}, \qquad Q^A = -\frac{F_{Av}}{8\pi F_v} \tag{2.92}$$

3 Detection of Impulsive Lightlike Signals

The current design of gravitational wave detectors is aimed at observing gravitational waves that establish oscillatory motion in the test particles. As an example of such waves can serve the gravitational waves from a spiraling neutron star system. The lightlike signals we are going to study here just change the relative position of the particles as a function of proper time / affine parameter but do not establish oscillatory motion.

To study gravitational waves we take two test particles and observe their geodesic deviation equation in presence of gravitational waves [5]. We take the space to be a vacuum space-time $\mathcal{M}^- \bigcup \mathcal{M}^+$, except on the null hypersurface $\mathcal{N}$. In general there is a unique way to solder the interior and exterior space-times, depending on the choice of the latter. We chose our domains to be flat as in 2.3, because due to the symmetries of Minkowski space a plane null hypersurface can carry various different energy-momentum and gravitational wave components [6]. This kind of setup allows us to study the more general case of impulsive gravitational waves and null matter.

3.1 Timelike Geodesics Crossing a Null Hypersurface

The integral curves of a vector field T^μ form a 2-space M_2 and are the timelike geodesics of the test particles (fig. 3.1). So, T^μ forms a unit timelike vector field

$$g_{\mu\nu} T^\mu T^\nu = -1 \tag{3.1}$$

and its parallel transport along the geodesic vanishes

$$\dot{T}^\mu \equiv T^\mu{}_{|\nu} T^\nu = 0. \tag{3.2}$$

The dot will denote covariant differentiation in the direction of T^μ for any tensor defined along the geodesic curves. Neighboring integral curves T^μ are connected by an orthogonal vector X^μ tangent to M_2. Thus

$$g_{\mu\nu} T^\mu X^\nu = 0, \tag{3.3}$$

but the covariant differentiation of X^μ in the direction of T^μ is not 0 (see eq. (A.2) of Appendix A)

$$\dot{X}^\mu = T^\mu{}_{|\nu} X^\nu = X^\mu{}_{|\nu} T^\nu \tag{3.4}$$

and the second differentiation will give the geodesic deviation equation (see eq. (A.4) of Appendix A)

$$\ddot{X}^\mu = -R^\mu{}_{\lambda\sigma\rho} T^\lambda X^\sigma T^\rho, \tag{3.5}$$

where $R^{\mu}{}_{\lambda\sigma\rho}$ are the components of the Riemann tensor in the space-time $\mathcal{M}^{-} \bigcup \mathcal{M}^{+}$.

Now let's discuss the behavior of the vectors T^{μ} and X^{μ} when the particles cross the hypersurface. Because the particles are massive, their geodesics are timelike, they can not move along the null hypersurface, so there will not be any displacement in the positions of the particles in the tangent directions to the hypersurface. So the vector X^{μ}, the geodesic deviation vector, will not have any jump. There will not be any jump in the velocities of the particles also. Although the Riemann curvature tensor contains a δ function and there is an impulsive signal on the hypersurface, the hypersurface itself doesn't effect the movement of the particle. The only effect that the components of the impulsive signal will have on the system is the change in the space-time of two domains, which depends on the junction conditions. So the values of the position and the velocities of a particle on the hypersurface, which reaches the hypersurface from the interior domain, will be initial conditions for the particle in the further movement in the exterior domain. The history of the particles on the hypersurface will be a smooth curve (fig. 3.1):

$$[T^{\mu}] = [X^{\mu}] = 0$$

We can also prove this by looking at the geodesic equation of a particle.

$$\dot{T}^{\mu} = -\Gamma^{\mu}_{\nu\sigma} T^{\nu} T^{\sigma}$$

If T^{μ} would have a jump across the hypersurface then it would contain a Heaviside step function θ. This means that the derivative of this vector in the left-hand side would contain a Dirac δ-function and because Christoffel symbols contain only θ function this would raise inconsistency between left-hand and right-hand sides of the geodesic equation.

So, we will only see a jump in the first derivatives of T^{μ} and X^{μ}, which are related to the acceleration of the particles. As the metrics are continuous and the derivatives of the metrics in the tangential direction to the hypersurface are also continuous, the jump in the acceleration of the particles can only take place in the transverse direction to the hypersurface (so, it must be proportional to the normal vector to the hypersurface, like the jump in $[g_{\mu\nu,\alpha}]$):

$$[T^{\mu}{}_{,\lambda}] = \eta P^{\mu} n_{\lambda}, \tag{3.6}$$

$$[X^{\mu}{}_{,\lambda}] = \eta W^{\mu} n_{\lambda}, \tag{3.7}$$

where P^{μ} and W^{μ} are defined on $\mathcal{N}$. A set of three vector fields is defined by parallel transporting e_a tangential vectors on the hypersurface $\mathcal{N}$, which will become three vectors E_a

$$\dot{E}^{\mu}_{a} = 0. \tag{3.8}$$

Of course in $\mathcal{N}$, $E_a = e_a$. Similar to the jumps of the vectors T^{μ} and X^{μ}, we define the jump of E_a across $\mathcal{N}$ as follows

$$[E^{\mu}_{a,\lambda}] = \eta F^{\mu}_{a} n_{\lambda}, \tag{3.9}$$

where F is defined on $\mathcal{N}$.

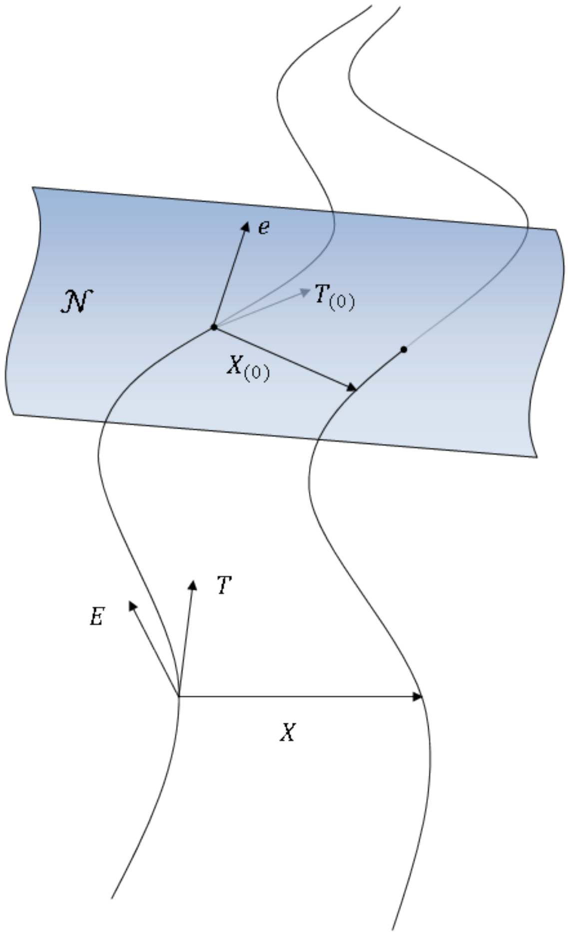

Figure 3.1: This diagram shows the geodesics of two particles crossing the hypersurface $\mathcal{N}$. T is the tangential vector to the geodesic curve, X is the deviation vector of the congruence of geodesics and E is defined as the parallel transport of the basis vectors on the hypersurface along the geodesic curve. There is no jump in the position or the velocity of the particles and only the first derivatives of the geodesic and deviation vectors have jumps.

The value of X^μ on $\mathcal{N}$ will be denoted by $X^\mu_{(0)}$. Similarly we will take T^μ to be $T^\mu_{(0)}$ on the hypersurface. We can express $X^\mu_{(0)}$ via $T^\mu_{(0)}$ and e^μ_a

$$X^\mu_{(0)} = X_{(0)} T^\mu_{(0)} + X^a_{(0)} e^\mu_a, \tag{3.10}$$

where $X_{(0)}$ and $X^a_{(0)}$ are some functions evaluated on $\mathcal{N}$.

We will derive some relations which are important for our discussion. First we will use equation (3.1) to write

$$N^\lambda [\partial_\lambda (g_{\mu\nu} T^\mu T^\nu)] = 0.$$

This means that T^μ stays timelike on both sides of the hypersurface. The left hand side of this equation can be modified using the fact that T^μ is continuous on $\mathcal{N}$

$$\begin{aligned} N^\lambda [\partial_\lambda (g_{\mu\nu} T^\mu T^\nu)] &= N^\lambda [\partial_\lambda g_{\mu\nu}] T^\mu_{(0)} T^\nu_{(0)} + N^\lambda g_{\mu\nu} [T^\mu{}_{,\lambda} T^\nu + T^\nu{}_{,\lambda} T^\mu] \\ &= \gamma_{\mu\nu} T^\mu_{(0)} T^\nu_{(0)} + 2\eta g_{\mu\nu} T^\nu_{(0)} P^\mu N^\lambda n_\lambda \\ &= \gamma_{\mu\nu} T^\mu_{(0)} T^\nu_{(0)} + 2 P_\mu T^\mu_{(0)} = 0 \end{aligned}$$

So we get

$$\gamma_{\mu\nu} T^{\mu}_{(0)} T^{\nu}_{(0)} + 2P_{\mu} T^{\mu}_{(0)} = 0 \tag{3.11}$$

The second equation that we will need can be derived from (3.2). Taking the jump of that equation, we can write

$$\begin{aligned}
\left[T^{\mu}{}_{|\nu} T^{\nu}\right] &= T^{\nu}_{(0)}[\partial_\nu T^{\mu} + \Gamma^{\mu}_{\nu\rho} T^{\rho}] = T^{\nu}_{(0)}[T^{\mu}{}_{,\nu}] + T^{\nu}_{(0)} T^{\rho}_{(0)} [\Gamma^{\mu}_{\nu\rho}] \\
&= \eta P^{\mu} T^{\nu}_{(0)} n_\nu + \eta T^{\nu}_{(0)} T^{\rho}_{(0)} (\gamma^{\mu}_{(\nu} n_{\rho)} - \frac{1}{2}\gamma_{\nu\rho} n^{\mu}) \\
&= \eta P^{\mu} T^{\nu}_{(0)} n_\nu - \frac{\eta}{2} T^{\nu}_{(0)} T^{\rho}_{(0)} \gamma_{\nu\rho} n^{\mu} + \frac{\eta}{2} T^{\nu}_{(0)} T^{\rho}_{(0)} (\gamma^{\mu}_{\nu} n_\rho + \gamma^{\mu}_{\rho} n_\nu) \\
&= 0
\end{aligned}$$

This can be rewritten as

$$\gamma_{\nu\rho} T^{\nu}_{(0)} T^{\rho}_{(0)} n^{\mu} = 2 T^{\nu}_{(0)} n_\nu (P^{\mu} + \gamma^{\mu}_{\nu} T^{\nu}_{(0)}). \tag{3.12}$$

Another useful equation can be obtained from (3.3).

$$\begin{aligned}
N^{\lambda}[\partial_\lambda (g_{\mu\nu} X^{\mu} T^{\nu})] &= N^{\lambda}[\partial_\lambda g_{\mu\nu}] X^{\mu}_{(0)} T^{\nu}_{(0)} + N^{\lambda} g_{\mu\nu} [X^{\mu}{}_{,\lambda} T^{\nu} + T^{\nu}{}_{,\lambda} X^{\mu}] \\
&= \gamma_{\mu\nu} X^{\mu}_{(0)} T^{\nu}_{(0)} + \eta N^{\lambda} n_\lambda g_{\mu\nu} (T^{\nu}_{(0)} W^{\mu} + X^{\mu}_{(0)} P^{\nu}) \\
&= \gamma_{\mu\nu} X^{\mu}_{(0)} T^{\nu}_{(0)} + T^{\nu}_{(0)} W_\nu + X^{\nu}_{(0)} P_\nu = 0
\end{aligned}$$

So that

$$\gamma_{\mu\nu} X^{\mu}_{(0)} T^{\nu}_{(0)} + T^{\nu}_{(0)} W_\nu + X^{\nu}_{(0)} P_\nu = 0. \tag{3.13}$$

Taking the jumps of the last two parts of the equation (3.4) we get

$$[T^{\mu}{}_{|\nu} X^{\nu}] = [T^{\mu}{}_{|\nu}] X^{\nu}_{(0)} = \eta P^{\mu} n_\nu X^{\nu}_{(0)} + [\Gamma^{\mu}_{\nu\lambda}] X^{\nu}_{(0)} T^{\lambda}_{(0)},$$

$$[X^{\mu}{}_{|\nu} T^{\nu}] = [X^{\mu}{}_{|\nu}] T^{\nu}_{(0)} = \eta W^{\mu} n_\nu T^{\nu}_{(0)} + [\Gamma^{\mu}_{\nu\lambda}] X^{\lambda}_{(0)} T^{\nu}_{(0)}.$$

These two must be equal, so

$$\begin{aligned}
P^{\mu} n_\nu X^{\nu}_{(0)} =& W^{\mu} n_\nu T^{\nu}_{(0)} = P^{\mu} n_\nu (X_{(0)} T^{\nu}_{(0)} + X^{a}_{(0)} e^{\nu}_{a}) \\
&\Longrightarrow W^{\mu} n_\nu T^{\nu}_{(0)} = P^{\mu} n_\nu X_{(0)} T^{\nu}_{(0)},
\end{aligned}$$

from which follows

$$W^{\mu} = X_{(0)} P^{\mu}. \tag{3.14}$$

Taking the jump of the equation (3.8) will yield another equation

$$(n_\lambda T^{\lambda}_{(0)}) F^{\mu}_{a} = -\frac{1}{2} \gamma^{\mu}_{\lambda} e^{\lambda}_{a} (n_\rho T^{\rho}_{(0)}) + \frac{1}{2} n^{\mu} (\gamma_{\mu\nu} e^{\mu}_{a} T^{\nu}_{(0)}). \tag{3.15}$$

The component of X^{μ} in the direction of T^{μ} is 0 and the components in the directions of E^{μ}_{a} are

$$X_a = g_{\mu\nu} X^{\mu} E^{\nu}. \tag{3.16}$$

Taking into account (3.10) we can write

$$X^{\mu} = X T^{\mu} + X^{a} E^{\mu}_{a}. \tag{3.17}$$

Multiplying this by T_μ and having in mind that X^μ and T^μ are orthogonal, we can write

$$X = X^a E^\mu_a T_\mu \tag{3.18}$$

Plugging this back into (3.17) and the obtained equation into (3.16) we will get

$$X_a = P_{ab} X^b, \tag{3.19}$$

where

$$P_{ab} = g_{\mu\nu} E^\mu_a E^\nu_b + E^\mu_a T_\mu E^\nu_b T_\nu \tag{3.20}$$

As E^μ_a and T^μ are parallel transported vectors along the geodesic curve, we can make a use of the conclusion in the Appendix C and write

$$P_{ab} = p_{ab} = g_{ab} + (T_{(0)\mu} e^\mu_a)(T_{(0)\nu} e^\nu_b) \tag{3.21}$$

and thus

$$X_a = p_{ab} X^b. \tag{3.22}$$

3.2 Geodesic Deviation Equation

Now we are going to discuss the behavior of the orthogonal connecting vector X^μ when it crosses the hypersurface $\mathcal{N}$. We will assume the particles move from the past ($u < 0$) to the future ($u > 0$). We will derive an equation for the projection of the geodesic deviation vector between two particles (3.16), taking into account the derivations made in the section 3.1. A derivation of this projection has already been suggested in [5] and is discussed in Appendix B of this thesis. But the calculations in this section differ from the one suggested in [5] with the advantages that it is more clear to see the use of the geodesic deviation equation and is more straightforward.

So, we will start from the geodesic deviation equation (see eq. (A.4) of Appendix A)

$$\ddot{X}^\mu = -R^\mu{}_{\lambda\sigma\rho} T^\lambda X^\sigma T^\rho,$$

where by dots we denoted the derivatives of X^μ along T^μ. Assuming the same decomposition (3.17)

$$X^\mu = X T^\mu + X^a E^\mu_a, \quad \text{with} \quad X = X^a E^\mu_a T_\mu$$

for the vector X^μ, we can write for its derivative along the geodesic vector T_μ

$$\mathrm{D_T} X^\mu = \mathrm{D_T}(X T^\mu + X^a E^\mu_a) = \left(\frac{\mathrm{d}X}{\mathrm{d}u}\right) T^\mu + \left(\frac{\mathrm{d}X^a}{\mathrm{d}u}\right) E^\mu_a,$$

where $\mathrm{D_T}$ is the derivative along T^μ (previously we used dotes instead)and u is the geodesic parameter. Here we used the facts that T^μ is the tangent vector along geodesic and E^μ_a is a parallel transport along T^μ, so

$$\mathrm{D_T} T^\mu = \mathrm{D_T} E^\mu_a = 0.$$

The second derivative of X^μ along T^μ will give

$$\mathrm{D_T}^2 X^\mu = \left(\frac{\mathrm{d}^2 X}{\mathrm{d}u^2}\right) T^\mu + \left(\frac{\mathrm{d}^2 X^a}{\mathrm{d}u^2}\right) E^\mu_a. \tag{3.23}$$

For the derivative of X we can write

$$\frac{\mathrm{d}X}{\mathrm{d}u} = \frac{\mathrm{d}X^b}{\mathrm{d}u} E_b^\alpha T_\alpha + X^a \frac{\mathrm{d}(E_a^\alpha T_\alpha)}{\mathrm{d}u} = \frac{\mathrm{d}X^b}{\mathrm{d}u} E_b^\alpha T_\alpha$$

The second term is 0 because both E_a^α and T^α preserve their norms and also the angle between them doesn't change along u. Taking the second derivative of X on u will yield

$$\frac{\mathrm{d}^2 X}{\mathrm{d}u^2} = \frac{\mathrm{d}^2 X^b}{\mathrm{d}u^2} E_b^\alpha T_\alpha$$

Plugging this into (3.23) we will find

$$\mathrm{D_T}^2 X^\mu = (E_a^\alpha T_\alpha T^\mu + E_a^\mu) \frac{\mathrm{d}^2 X^a}{\mathrm{d}u^2}$$

Using the value of p_{ab} from (3.21) we find for the second derivative of X^μ along T^μ

$$g_{\mu\nu} E_b^\nu (\mathrm{D_T}^2 X^\mu) = \big(g_{\mu\nu} E_b^\nu E_a^\mu + (E_a^\alpha T_\alpha)(E_b^\nu T_\mu)\big) \frac{\mathrm{d}^2 X^a}{\mathrm{d}u^2} = p_{ab} \frac{\mathrm{d}^2 X^a}{\mathrm{d}u^2} = \frac{\mathrm{d}^2 X_b}{\mathrm{d}u^2}.$$

Summarizing this we finally find a connection between the derivative of X^μ along T^μ and its projections onto vectors E_a^μ

$$g_{\mu\nu} E_a^\nu (\mathrm{D_T}^2 X^\mu) = \frac{\mathrm{d}^2 X_a}{\mathrm{d}u^2}.$$

Plugging the geodesic deviation equation (3.5) into the upper relation will yield

$$\frac{\mathrm{d}^2 X_a}{\mathrm{d}u^2} = -g_{\mu\nu} E_a^\nu R^\mu{}_{\lambda\sigma\rho} T^\lambda X^\sigma T^\rho.$$

Using the decomposition (3.17) for X^μ, we can modify the last equation

$$\frac{\mathrm{d}^2 X_a}{\mathrm{d}u^2} = -g_{\mu\nu} E_a^\nu \big(X R^\mu{}_{\lambda\sigma\rho} T^\lambda T^\sigma T^\rho - R^\mu{}_{\lambda\sigma\rho} T^\lambda (X^b E_b^\sigma) T^\rho\big).$$

The first term here is 0, as $R^\mu{}_{\lambda\sigma\rho}$ is antisymmetric on σ and ρ indexes, so we are left with

$$\frac{\mathrm{d}^2 X_a}{\mathrm{d}u^2} = -g_{\mu\nu} E_a^\nu R^\mu{}_{\lambda\sigma\rho} T^\lambda (X^b E_b^\sigma) T^\rho$$

Because $R^\mu{}_{\lambda\sigma\rho}$ has a term proportional to $\delta(u)$ according to the equation (2.37), the second derivative of X_a will also contain such a term

$$\frac{\mathrm{d}^2 X_a}{\mathrm{d}u^2} = -E_a^\nu \big(\tilde{R}_{\nu\lambda\sigma\rho} + \hat{R}_{\nu\lambda\sigma\rho} \eta\chi\delta(u)\big) T^\lambda (X^b E_b^\sigma) T^\rho.$$

What we are interested is the jump in the first derivative of X_a. To find it we will integrate the last equation and take its jump

$$\left[\frac{\mathrm{d}X_a}{\mathrm{d}u}\right] = \left[\int \left(\frac{\mathrm{d}^2 X_a}{\mathrm{d}u^2}\right) \mathrm{d}u\right]$$
$$= -\left[\int \left(E_a^\nu \tilde{R}_{\nu\lambda\sigma\rho} T^\lambda (X^b E_b^\sigma) T^\rho + \eta\chi\delta(u) E_a^\nu \hat{R}_{\nu\lambda\sigma\rho} T^\lambda (X^b E_b^\sigma) T^\rho\right) \mathrm{d}u\right].$$

The first term under integral can be expended into Taylor series, so its integral will only contain terms proportional to powers of u. The jumps of such terms will certainly be 0. The second term under integral contains a Dirac delta function of the parameter u, so its integral will be

$$\left[\frac{\mathrm{d}X_a}{\mathrm{d}u}\right] = -\left[\int \left(\eta\chi\delta(u)E^\nu_a \hat{R}_{\nu\lambda\sigma\rho}T^\lambda(X^a E^\sigma_a)T^\rho\right)\mathrm{d}u\right]$$
$$= -\eta\chi e^\nu_a (X^b_{(0)}e^\sigma_b)\hat{R}_{\nu\lambda\sigma\rho}T^\lambda_{(0)}T^\rho_{(0)}.$$

The singular term of the Riemann tensor $\hat{R}_{\nu\lambda\sigma\rho}$ is given by (2.40). It contains four terms with n normal vector with different indexes. If we replace $\hat{R}_{\nu\lambda\sigma\rho}$ in the last jump equation with that relation, most of the these n terms will vanish due to the sum with e^ν_a and e^σ_b terms, so we will get

$$\left[\frac{\mathrm{d}X_a}{\mathrm{d}u}\right] = -\eta\chi e^\nu_a (X^b_{(0)}e^\sigma_b)(-\frac{1}{2}n_\lambda\gamma_{\nu\sigma}n_\rho)T^\lambda_{(0)}T^\rho_{(0)} = \frac{1}{2}\eta\chi^{-1}\gamma_{\nu\sigma}e^\nu_a e^\sigma_b X^b_{(0)},$$

so we find

$$\left[\frac{\mathrm{d}X_a}{\mathrm{d}u}\right] = \frac{1}{2}\,\eta\,\chi^{-1}\,\gamma_{ab}X^b_{(0)}. \tag{3.24}$$

To find X_a for positive u we need to integrate (3.24) again:

$$X_a - X_{(0)a} = u\left.\frac{\mathrm{d}\bar{X}_a}{\mathrm{d}u}\right|_{u=0} + \frac{1}{2}\,\eta\,\chi^{-1}\,u\,\gamma_{ab}X^b_{(0)}.$$

This can be easily transformed into the following form

$$X_a = \left(p_{ab} + \frac{1}{2}\eta\chi^{-1}u\gamma_{ab}\right)X^b_{(0)} + u\,\bar{V}_{(0)a}$$

3.2.1 Wave and Shell Parts of the Signal

As it was mentioned in the introduction, an impulsive lightlike signal can be either a null shell or an impulsive gravitational wave, or a mixture of both. Let's consider the intrinsic form (2.69) of the surface stress-energy tensor.

$$16\pi S^{ab} = -(\gamma_{cd}g^{cd}_*)n^a n^b - (\gamma_{cd}n^c n^d)g^{ab}_* + (g^{ac}_* n^b n^d + g^{bc}_* n^a n^d)\gamma_{cd}.$$

The symmetric tensor γ_{ab}, from which this stress-energy tensor is constructed, has six independent components. We can write γ_{ab} as a sum of two different matrices

$$\gamma_{ab} = \hat{\gamma}_{ab} + \bar{\gamma}_{ab} \tag{3.25}$$

where $\hat{\gamma}_{ab}$ doesn't contribute to the stress-energy tensor. For this to be true it should satisfy the following conditions

$$\hat{\gamma}_{ab}n^b = 0, \qquad g^{ab}_*\hat{\gamma}_{ab} = 0. \tag{3.26}$$

Thus, $\hat{\gamma}_{ab}$ has two independent components. This conditions are independent of the freedom of choice of g^{ab}_*. $\hat{\gamma}_{ab}$ can be explicitly expressed the following way

$$\hat{\gamma}_{ab} = \gamma_{ab} - \frac{1}{2}g^{cd}_*\gamma_{cd}g_{ab} - 2\eta\gamma_{(a}N_{b)} + \eta^2\gamma^\dagger\{N_a N_b - \frac{1}{2}(N\cdot N)g_{ab}\}. \tag{3.27}$$

While the four components of $\bar{\gamma}_{ab}$ contribute to the stress-energy tensor, the two components of $\hat{\gamma}_{ab}$ are required to construct the two degrees of freedom of polarization present in an impulsive gravitational wave.

We will use decomposition (3.25) to study the effects of wave and shell parts of the lightlike signal. We will chose as basis vectors

$$e_1^\mu = n_1^\mu \text{ and } e_A$$

where e_A is orthogonal to $T^\mu_{(0)}$, with $A = 2, 3$. Using these values and equations (2.57) and (2.59) we can calculate

$$e_1^\mu = n^\mu = n^a e^\mu_{(a)} = n^1 e_1^\mu + n^2 e_2^\mu + n^3 e_3^\mu \implies n^a = \delta^a_1$$

and

$$g_{ab} n^b = 0 = g_{ab} \delta^b_1 = g_{a1} \implies g_{a1} = g_{1a} = 0.$$

To define N^μ we assume that

$$N^\mu = T^\mu_{(0)}, \quad N \cdot e_{(A)} = 0, \quad N \cdot n = \eta^{-1}$$

Hence we get

$$\begin{aligned} N^\mu N_\mu &= T^\mu T_\mu = -1 \\ N_\mu e^\mu_{(A)} &= N_A \quad = 0 \\ N_1 &= N_\mu e_1^\mu = N^\mu n_\mu = \eta^{-1}. \end{aligned}$$

From equations (2.67) and (2.68) follows

$$g_*^{A1} = 0, \quad g_*^{11} = \eta^2, \quad g_*^{AB} = g^{AB} \tag{3.28}$$

We will take the common surface of e_2 and e_3 vectors to be the signal front in $\mathcal{N}$. We will also assume that at the moment when signal arrives the test particles are in that $(2,3)$–2-surface; Since this 2-surface is a Riemannian 2-surface, we can chose the coordinates $\{x^A\}$ in such a way that g_{AB} is diagonal so

$$g_{AB} = p^{-2} \delta_{AB},$$

where $p = p(x^A)$ is some function. It follows from (3.28) that g_*^{ab} in matrix form is

$$g_* = \begin{pmatrix} \eta^2 & 0 & 0 \\ 0 & p^2 & 0 \\ 0 & 0 & p^2 \end{pmatrix}$$

We will denote the tensor from (B.12) by $\tilde{p}_{ab}$ and calculate it to be

$$\tilde{p}_{ab} = \begin{pmatrix} \eta^{-2} & 0 & 0 \\ 0 & p^{-2} & 0 \\ 0 & 0 & p^{-2} \end{pmatrix}$$

Using (3.26), we can find the components of $\hat{\gamma}_{ab}$.

$$\hat{\gamma}_{ab} n^b = 0 = \hat{\gamma}_{ab} \delta^b_1 = \hat{\gamma}_{a1} \implies \hat{\gamma}_{a1} = \hat{\gamma}_{1a} = 0,$$

$$g_*^{ab}\hat{\gamma}_{ab} = 0 = g_*^{22}\hat{\gamma}_{22} + g_*^{33}\hat{\gamma}_{33} = p^{-2}(\hat{\gamma}_{22} + \hat{\gamma}_{33}) \implies \hat{\gamma}_{22} = -\gamma_{33}.$$

So we can write in matrix form

$$\hat{\gamma}_{AB} = \begin{pmatrix} \hat{\gamma}_{22} & \hat{\gamma}_{23} \\ \hat{\gamma}_{23} & -\hat{\gamma}_{22} \end{pmatrix}$$

In the special frame we've chosen, the stress-energy tensor (2.69) will have the following components

$$16\pi S^{ab} = -\eta^2\gamma_{11}\delta_1^a\delta_1^b - p^2(\gamma_{22} + \gamma_{33})\delta_1^a\delta_1^b - \gamma_{11}g_*^{ab} + (g_*^{ac}\delta_1^b\delta_1^d + g_*^{bc}\delta_1^a\delta_1^d)\gamma_{cd},$$

so

$$\begin{aligned} 16\pi S^{11} &= -p^2(\gamma_{22} + \gamma_{33}), \\ 16\pi S^{1B} &= p^2\gamma_{1B}, \\ 16\pi S^{AB} &= -p^2\gamma_{11}\delta_{AB}. \end{aligned}$$

Using (B.12) we can find the projections of the connecting vector to be

$$\eta^{-2}X^1 = X_1 = \eta^{-2}X^1_{(0)} + \frac{u}{2}\eta\chi^{-1}\gamma_{1b}X^b_{(0)} + u\,\bar{V}_{(0)1}, \tag{3.29}$$

$$\eta^{-2}X^A = X_A = \eta^{-2}X^A_{(0)} + \frac{u}{2}\eta\chi^{-1}\gamma_{Ab}X^b_{(0)} + u\,\bar{V}_{(0)A}. \tag{3.30}$$

A few remarks can be made about these equations.

First of all, from equation (3.29) we can see that *the wave component of the signal doesn't contribute to* X^1 as the only components of γ_{ab} involved in that equation are γ_{1b}, whose $\hat{\gamma}_{ab}$ composites are 0.

Then, let's consider the case when test particles' movements are bounded to the signal front $(2, 3)$–2-suface and their initial relative velocity in the third direction of $\mathcal{N}$ is 0. So

$$X^1_{(0)} = \bar{V}_{(0)1} = 0$$

We can rewrite (3.29) the following way

$$\eta^{-2}X^1 = \frac{u}{2}\eta\chi^{-1}\gamma_{1B}X^B_{(0)}.$$

Also, for S^{1B} we can write

$$16\pi S^{1B} = p^2\gamma_{1B} = 16\pi Q^B,$$

where Q^a is the surface energy currents given by the equation (2.75). Combining the last two equations we can conclude that if $S^{1B} \neq 0$, which means that *if there is surface energy current in the lightlike shell then* $X^1 \neq 0$, *so the particles initially moving in the signal front are displaced out of the 2-surface after encountering the signal.*

Now suppose that there is no surface energy current, so $S^{1B} = 0$. Also we will assume that the test particles are initially at rest ($X^1_{(0)} = \bar{V}_{(0)b} = 0$). From (3.29) we will get that $X^1 = 0$. We can write (3.30) for small u, using the equation for S^{11} and the decomposition of γ_{ab}, in the following form

$$X^A = (1 - 4\pi\,u\,\eta\,\chi^{-1}\,S^{11})(\delta_{AB} + \frac{u}{2}\,\eta\,\chi^{-1}\,p^2\,\hat{\gamma}_{AB})X^B_{(0)}. \tag{3.31}$$

The factor $\delta_{AB}+\frac{u}{2}\,\eta\,\chi^{-1}\,p^2\,\hat{\gamma}_{AB}$, as it depends on $\hat{\gamma}_{AB}$ describes the distortion effect of the wave part of the signal on the test particles. The other factor $1-4\pi\,u\,\eta\,\chi^{-1}\,S^{11}<1$ is related to the lightlike shell and has focusing effect. So we can conclude that *if the lightlike shell has no surface energy current in parallel with a gravitational wave, the effect of the signal on the test particles at rest in the signal front is a displacement relative to each other and a distortion due to the gravitational wave diminished by the presence of the lightlike shell.*

3.2.2 An Example of Plane Fronted Lightlike Signal

Let's consider the application of (3.31) on a plane fronted lightlike signal described in subsection 2.3. There we've discussed a case of two flat domains $\mathcal{M}^+$ and $\mathcal{M}^-$ with metrics

$$\mathrm{d}s_\pm^2 = -2\,\mathrm{d}u_\pm\,\mathrm{d}v_\pm + \mathrm{d}x_\pm^2 + \mathrm{d}y_\pm^2.$$

and junction conditions

$$x_+ = x_-, \qquad y_+ = y_-, \qquad v_+ = F(v_-, x_-, y_-), \tag{3.32}$$

on the null hypersurface $\mathcal{N}$. We will assume the function $F(v_-, x_-, y_-)$ to have the following explicit form

$$F(v,x,y) = v - \frac{a}{2}(x^2+y^2) + \frac{b}{2}(x^2-y^2) + cxy, \tag{3.33}$$

where, as in subsection 2.3, we have made the following notations

$$u_- = u, \qquad v_- = v, \qquad x_- = x, \qquad y_- = y.$$

In equation (3.33) $a, b,$ and c are constants and $a > 0$.

As in the previous section, we will choose the normal n to coincide with e_1, where we choose $e^\mu_a = \delta^\mu_a$, so

$$n^\mu = e^\mu_1 = \delta^\mu_1.$$

We can find n^a using this condition.

$$n^\mu = e^\mu_1 = n^a e^\mu_a \implies n^a = \delta^a_1 \tag{3.34}$$

But unlike that section, here we will choose the transversal vector N to be null, so

$$N\cdot N = 0, \quad N\cdot e_{(A)} = 0, \quad N\cdot n = -1$$

It follows that we choose $\eta = -1$. From these conditions we can find N_a vector:

$$\begin{aligned} N_\mu n^\mu &= -1 = N_\mu e^\mu_1 = N_1 \\ N_\mu e^\mu_{(A)} &= \;\;0\;\; = N_A. \end{aligned}$$

So,

$$N_a = -\delta^1_a. \tag{3.35}$$

One of the defining conditions of g_*^{ab} is equation (2.68)

$$g_*^{ab} N_b + \eta n^a (N \cdot N) = 0,$$

where the second term is clearly 0 in our case and we find $g_*^{ab} N_b = 0 = -g_*^{ab} \delta_a^1 = -g_*^{1b}$, from which follows

$$g_*^{1a} = 0.$$

Similarly from (2.67) it follows $g_*^{AB} = g^{AB}$, which is the inverse of g_{AB}. Taking a specific choice for g_{ab}, we can write

$$g_{ab} = g_*^{ab} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So p in equation (3.31) is equal to 1. The choice of N^μ we made here is similar to the one made in (2.91) so we can similarly write

$$\gamma_{ab} = \frac{2F_{ab}}{F_v}$$

where, the subscripts denote partial derivatives on F. Using this equation we can find

$$\gamma_{ab} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2(-a+b) & 2c \\ 0 & 2c & 2(-a-b) \end{pmatrix}.$$

Using the equation (3.27) for $\hat{\gamma}$ we can find

$$\hat{\gamma}_{ab} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2b & 2c \\ 0 & 2c & -2b \end{pmatrix}. \qquad (3.36)$$

Using the expressions for γ_{ab} and g_*^{ab} we can calculate the components (2.75) of S^{ab} to be

$$\begin{aligned} P &= -\frac{1}{16\pi}\gamma^\dagger = -\frac{1}{16\pi}\gamma_{ab} n^a n^b = \gamma_{11} = 0 \\ Q^a &= \frac{1}{16\pi} g_*^{ac} \gamma_c = \frac{1}{16\pi} g_*^{ac} \gamma_\mu e^\mu_{(c)} = 0 \\ \mu &= -\frac{1}{16\pi}\gamma_{cd} g_*^{cd} = -\frac{1}{16\pi}(\gamma_{22} + \gamma_{33}) = \frac{a}{4\pi} \end{aligned}$$

We see that only the energy density is non-zero so we can write

$$S^{ab} = \mu n^a n^b \implies S^{ab} = 0 \quad \text{except} \quad S^{11} = \mu \qquad (3.37)$$

To derive the forms for S^{11} and $\hat{\gamma}_{ab}$, we have made all the definitions the same way as in the previous subsection, except for the definition of N^μ. This difference, of course, lead us to a different form of g_*^{ab} compared to the one derived in the last subsection. But as S^{11} and $\hat{\gamma}_{ab}$ are intrinsic matrices, they are independent of the choice of N^μ and g_*^{ab} and, thus, as we

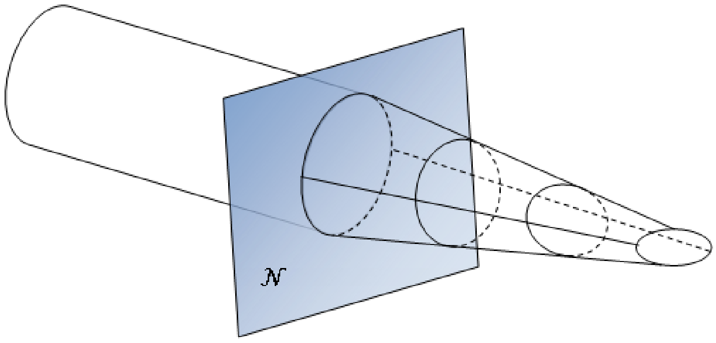

Figure 3.2: This diagram shows the evolution of the history of the particles which were initially arranged in a circular orbit. The junction conditions for this specific case are given in equation (3.32) and the domains on each side of the hypersurface are taken to be flat spaces. After crossing the hypersurface, the orbit of the particles deforms into an ellipse and rotated through a small angle.

can see they stayed unchanged. This also means that the equation (3.31) is applicable independent of the choice of N^μ. So, for X^2 and X^3 we get

$$X^2 = (1 - 2\,a\,u)\left[(1 + u\,b)X^2_{(0)} + u\,c\,X^3_{(0)}\right], \tag{3.38}$$

$$X^3 = (1 - 2\,a\,u)\left[u\,c\,X^2_{(0)} + (1 - u\,b)X^3_{(0)}\right]. \tag{3.39}$$

As we are discussing the case of small u we can modify these equations the following way (as we can ignore the $\sim u^2$ terms)

$$X^2 = (1 - 2\,a\,u)(1 + b\,u)\left(X^2_{(0)} + c\,u\,X^3_{(0)}\right), \tag{3.40}$$

$$X^3 = (1 - 2\,a\,u)(1 - b\,u)\left(c\,u\,X^2_{(0)} + X^3_{(0)}\right). \tag{3.41}$$

Hence if we have a circle $(X^2_{(0)})^2 + (X^3_{(0)})^2 = const$ of particles then after encountering the lightlike signal they will 1) undergo a rotation through a small angle cu and 2) a deformation into an ellipse with semi axes of lengths $(1 - 2\,a\,u)(1 + b\,u)$ and $(1 - 2\,a\,u)(1 - b\,u)$.

3.3 Lightlike Geodesics Crossing a Null Hypersurface

In this section we will discuss the case of lightlike particles crossing the null hypersurface $\mathcal{N}$. This case has not been discussed before and as it is explained in the following sections, the resulting equation for the projection of the geodesic deviation vector will contain noticeable and characteristic differences compared to the timelike case. As in the case of timelike geodesics we will denote the tangent vector to the geodesic by T^μ and the geodesic deviation vector by X^μ. $n^\mu_{(0)}$ will represent the normal vector to the hypersurface (we will occasionally refer to the same vector by e^α_1) and e^μ_A, where $A = 2, 3$, will represent the tangent vectors to the hypersurface.

As we have a freedom of choice of X^μ in the direction of T^μ, we will assume that these vectors are orthogonal. To impose this orthogonality condition we need to use the following projection tensor

$$h_{\alpha\beta} = g_{\alpha\beta} + T_\alpha N_\beta + N_\alpha T_\beta,$$

where N^μ is the parallel transport of the null transversal vector to the hypersurface $N^\alpha_{(0)}$ and has the following relation with $T^\alpha_{(0)}$ (the geodesic vector on the hypersurface)

$$T^\alpha_{(0)} N_{(0)\alpha} = \eta^{-1},$$

where η is some constant. The tensor $h_{\alpha\beta}$ is actually the metric of the hypersurface orthogonal to both T^μ and N^μ. If we multiply this tensor with the geodesic deviation vector, we will get the latter's projection onto that hypersurface.

$$h^\alpha_\beta X^\beta = X^\alpha + N^\alpha (T_\lambda X^\lambda) + T^\alpha (N_\lambda X^\lambda)$$

If we want the vector X^α to be in that hypersurface (orthogonal to both N^α and T^α vectors), than the following equality should hold

$$N^\alpha (T_\lambda X^\lambda) + T^\alpha (N_\lambda X^\lambda) = 0$$

If we multiply this relation by T_α, we can find that $T_\alpha X^\alpha$ is restricted to be 0. Thus, we can see from the same relation, that $N_\alpha X^\alpha$ must also be 0.

We will take the vectors $\{N^\alpha_{(0)}, e^\alpha_A, n^\alpha_{(0)}\}$ as the basis vectors and construct the geodesic and geodesic deviation vectors from this basis:

$$X^\alpha_{(0)} = \alpha_{(0)x} N^\alpha_{(0)} + X^A_{(0)} e^\alpha_A + X^1_{(0)} n^\alpha_{(0)} \tag{3.42}$$

$$T^\alpha_{(0)} = \alpha_{(0)T} N^\alpha_{(0)} + T^A_{(0)} e^\alpha_A + T^1_{(0)} n^\alpha_{(0)} \tag{3.43}$$

As it is shown in the Appendix C, the product of any two vectors which have been parallel transported along T^α will neither have a jump term or a term proportional to the parameter u (geodesic vector parameter). That product will retain the value it has on the hypersurface $\mathcal{N}$. So, as a summary of the assumptions and calculations we have done so far, we can write:

$$\begin{array}{lll} N^\alpha N_\alpha = 0, & n_\alpha E^\alpha_a = 0, & T^\alpha X_\alpha = 0 \\ T^\alpha T_\alpha = 0, & N_\alpha E^\alpha_a = \lambda_a, & N^\alpha X_\alpha = 0 \\ T^\alpha N_\alpha = \eta^{-1}, & & \end{array} \tag{3.44}$$

$$X^\alpha = \alpha_x N^\alpha + X^A E^\alpha_a \tag{3.45}$$

$$T^\alpha = \alpha_T N^\alpha + T^A E^\alpha_a \tag{3.46}$$

In these equations $n^\alpha = E^\alpha_1$ and E^α_A are the parallel transports of the $n^\alpha_{(0)} = e^\alpha_1$ and e^α_A vectors along T^α respectively. It's clear that so far only two conditions have been imposed to define N^α. For complete definition we need to impose two other conditions, which is done through any of the two parameters of λ_a $(a = 1, 2, 3)$.

The jumps of the first derivatives of T^α, X^α and E^α_a vectors are still directed along the normal vector to $\mathcal{N}$ and can be defined by the same equations as in the case of timelike geodesic.

$$[T^\alpha{}_{,\lambda}] = \eta P^\alpha n_\lambda,$$
$$[X^\alpha{}_{,\lambda}] = \eta W^\alpha n_\lambda,$$
$$[E^\alpha_{a,\lambda}] = \eta F^\alpha_a n_\lambda,$$

We should stress that η in this equations is defined by $T^\alpha N_\alpha$, while in the case of timelike geodesics it was defined via $n^\alpha N_\alpha$. In the current case the $n^\alpha N_\alpha$ product may be chosen to be 0 which would lead to undesired consequences if the definition wasn't changed.

Also, the same equations as in the timelike geodesic case can be obtained for the jump vectors P^α and F^α_a

$$P^\mu = -\gamma^\mu_\nu T^\nu_{(0)} + n^\mu_{(0)} \frac{(\gamma_{\nu\rho} T^\nu_{(0)} T^\rho_{(0)})}{2(T^\nu_{(0)} n_{(0)\nu})},$$

$$F^\mu_a = -\frac{1}{2}\gamma^\mu_\lambda e^\lambda_a + n^\mu_{(0)} \frac{(\gamma_{\mu\nu} e^\mu_a T^\nu_{(0)})}{2(T^\nu_{(0)} n_{(0)\nu})}.$$

We can show that the jump vector W^α is proportional to the P^α vector but with a different proportionality factor than in the timelike case. For the geodesic deviation vector we have

$$[T^\mu{}_{|\nu} X^\nu] = [X^\mu{}_{|\nu} T^\nu]$$

and we can also find

$$[T^\mu{}_{|\nu} X^\nu] = [T^\mu{}_{|\nu}] X^\nu_{(0)} = \eta P^\mu (n_{(0)\nu} X^\nu_{(0)}) + [\Gamma^\mu_{\nu\lambda}] X^\nu_{(0)} T^\lambda_{(0)},$$
$$[X^\mu{}_{|\nu} T^\nu] = [X^\mu{}_{|\nu}] T^\nu_{(0)} = \eta W^\mu (n_{(0)\nu} T^\nu_{(0)}) + [\Gamma^\mu_{\nu\lambda}] X^\lambda_{(0)} T^\nu_{(0)}.$$

It follows from the last three equations that

$$W^\mu = P^\mu \frac{(n_{(0)\nu} X^\nu_{(0)})}{(n_{(0)\nu} T^\nu_{(0)})} = P^\mu \frac{\alpha_{(0)x}}{\alpha_{(0)T}}.$$

Having equations derived for jump vectors W^α, P^α and F^α_a we can find the projection X_a of the geodesic deviation vector on the hypersurface $\mathcal{N}$. We will assume that, as the particles cross the hypersurface, the change in the vectors X^α and E^α_a and in the metric $g_{\alpha\beta}$ is linear on u (for $u > 0$), like in (B.6)

$$X^\mu = \bar{X}^\mu + \eta\,\chi^{-1}\,u\,W^\mu$$

We will also assume linear approximation on u in tensors X_a, $\bar{X}^\mu$, $\bar{E}^\mu_a$ and $\bar{g}_{\mu\nu}$. After these assumptions, we can write

$$\begin{aligned} X_a &= g_{\mu\nu} X^\mu E^\nu_a \\ &= (\bar{g}_{\mu\nu} + \eta\,\chi^{-1}\,u\,\gamma^{\mu\nu})(\bar{X}^\mu + \eta\,\chi^{-1}\,u\,W^\mu)(\bar{E}^\nu_a + \eta\,\chi^{-1}\,u\,F^\nu_a) \\ &= \bar{g}_{\mu\nu}\bar{X}^\mu\bar{E}^\nu_a + \eta\,\chi^{-1}\,u\,(\gamma_{\mu\nu}\bar{X}^\mu\bar{E}^\nu_a + \bar{g}_{\mu\nu}W^\mu\bar{E}^\nu_a + \bar{g}_{\mu\nu}\bar{X}^\mu F^\nu_a) \\ &= \bar{g}_{\mu\nu}\bar{X}^\mu\bar{E}^\nu_a + \eta\,\chi^{-1}\,u\,(\gamma_{\mu\nu}X^\mu_{(0)}e^\nu_a + g^{(0)}_{\mu\nu}W^\mu e^\nu_a + g^{(0)}_{\mu\nu}X^\mu_{(0)}F^\nu_a) \end{aligned} \tag{3.47}$$

The linear term in parenthesis can be modified (having in mind that $X^{\mu}_{(0)}n_{(0)\mu} = \lambda_1\alpha_{(0)x}$, $T^{\mu}_{(0)}n_{(0)\mu} = \lambda_1\alpha_{(0)T}$ and $\alpha_{(0)x} = -\frac{X^a E^{\alpha}_a T_{\alpha}}{\eta^{-1}}$)

$$
\begin{aligned}
(\ldots) &= \gamma_{\mu\nu}X^{\mu}_{(0)}e^{\nu}_a + g^{(0)}_{\mu\nu}e^{\nu}_a\frac{\alpha_{(0)x}}{\alpha_{(0)T}}\left(-\gamma^{\mu}_{\rho}T^{\rho}_{(0)} + n^{\mu}_{(0)}\frac{(\gamma_{\rho\lambda}T^{\rho}_{(0)}T^{\lambda}_{(0)})}{2(T^{\alpha}_{(0)}n_{(0)\alpha})}\right) \\
&\qquad + g^{(0)}_{\mu\nu}X^{\mu}_{(0)}\left(-\frac{1}{2}\gamma^{\nu}_{\rho}e^{\rho}_a + n^{\nu}_{(0)}\frac{(\gamma_{\rho\lambda}e^{\rho}_a T^{\lambda}_{(0)})}{2(T^{\alpha}_{(0)}n_{(0)\alpha})}\right) \\
&= \frac{1}{2}\gamma_{\mu\nu}X^{\mu}_{(0)}e^{\nu}_a - \frac{1}{2}\frac{\alpha_{(0)x}}{\alpha_{(0)T}}\gamma_{\mu\nu}e^{\mu}_a T^{\nu}_{(0)} = \frac{1}{2}\gamma_{\mu\nu}e^{\nu}_a\left(X^{\mu}_{(0)} - \frac{\alpha_{(0)x}}{\alpha_{(0)T}}T^{\mu}_{(0)}\right) \\
&= \frac{1}{2}\gamma_{\mu\nu}e^{\nu}_a\left(X^{\mu}_{(0)} - \alpha_{(0)x}N^{\mu}_{(0)} - \frac{\alpha_{(0)x}}{\alpha_{(0)T}}T^{c}_{(0)}e^{\mu}_c\right) \\
&= \frac{1}{2}\gamma_{\mu\nu}e^{\nu}_a\left(X^{b}_{(0)}e^{\mu}_b + (X^{b}_{(0)}e^{\alpha}_b T_{(0)\alpha})\frac{T^{c}_{(0)}e^{\mu}_c}{\eta^{-1}\,\alpha_{(0)T}}\right) \\
&= \frac{1}{2}\left(\gamma_{ab} + (e^{\alpha}_b T_{(0)\alpha})\frac{T^{c}_{(0)}\gamma_{ac}}{\eta^{-1}\,\alpha_{(0)T}}\right)X^{b}_{(0)}
\end{aligned} \tag{3.48}
$$

For the first term in X_a (the term with bar tensors) we have

$$
\begin{aligned}
\bar{g}_{\mu\nu}\bar{X}^{\mu}\bar{E}^{\nu}_a &= g^{(0)}_{\mu\nu}X^{\mu}_{(0)}e^{\nu}_a + u\left.\frac{\mathrm{d}\bar{X}_a}{\mathrm{d}u}\right|_{u=0} \\
&= \alpha_{(0)x}(N_{(0)\alpha}e^{\alpha}_a) + X^{b}_{(0)}g^{(0)}_{\mu\nu}e^{\mu}_b e^{\nu}_b + u\left.\frac{\mathrm{d}\bar{X}_a}{\mathrm{d}u}\right|_{u=0} \\
&= \left[-\frac{\lambda_a}{\eta^{-1}}(e^{\alpha}_b T_{(0)\alpha}) + g_{ab}\right]X^{b}_{(0)} + u\left.\frac{\mathrm{d}\bar{X}_a}{\mathrm{d}u}\right|_{u=0}
\end{aligned} \tag{3.49}
$$

Plugging the decomposition (3.45) into the equation for X_a we can find

$$
\begin{aligned}
X_a = g_{\mu\nu}X^{\mu}E^{\nu}_a &= \alpha_x(N_{\alpha}E^{\alpha}_a) + X^b g_{\mu\nu}E^{\mu}_b E^{\nu}_b \\
&= \left[-\frac{\lambda_a}{\eta^{-1}}(e^{\alpha}_b T_{(0)\alpha}) + g_{ab}\right]X^b
\end{aligned} \tag{3.50}
$$

Using equations (3.47) to (3.50) we can finally write

$$
\begin{aligned}
X_a &= q_{ab}X^b \\
&= \left[q_{ab} + \frac{1}{2}\,\eta\,\chi^{-1}\,u\,\left(\gamma_{ab} + (e^{\alpha}_b T_{(0)\alpha})\frac{T^{c}_{(0)}\gamma_{ac}}{\eta^{-1}\,\alpha_{(0)T}}\right)\right]X^{b}_{(0)} + u\bar{V}_{(0)a}
\end{aligned} \tag{3.51}
$$

where $\bar{V}_{(0)a} = \left.\frac{\mathrm{d}\bar{X}_a}{\mathrm{d}u}\right|_{u=0}$ and represents the relative velocity of the particles on the hypersurface $\mathcal{N}$ and q_{ab} $(a = 1, 2, 3,\ A = 2, 3)$ is given by

$$
q_{ab} = -\frac{\lambda_a}{\eta^{-1}}(T_{(0)\alpha}\,e^{\alpha}_b) + g_{ab} = -\frac{\lambda_a}{\eta^{-1}}(\alpha_{(0)T}\lambda_b + T^A g_{Ab}) + g_{ab} \tag{3.52}
$$

3.4 Behavior of Timelike And Lightlike Geodesics

In this section we will discuss a specific choice of the null geodesic vector and will find out that a massless particle with that choice behaves much like a massive particle discussed in the section 3.2.1. The geodesic vector we are talking about is quite trivial:

$$T_\alpha E^\alpha_A = 0 \qquad T_\alpha n^\alpha = \sigma^{-1},$$

where σ^{-1} can be anything, except of course 0. These three relations along with the requirement that the T^α vector is null, define it completely. We should mention that so far we didn't make a specific choice of the vector N^α, and therefore, we still have two freedoms of choice in the parameters λ_a. As a result, we can't find the parameters α_T and T^a, unless we define λ_a. Now we will find relations between the parameters α_T, T^a and λ_a. We can write

$$T_\alpha n^\alpha = \sigma^{-1} = \alpha_T (N^\alpha n_\alpha) \qquad \Longrightarrow \alpha_T = \frac{\sigma^{-1}}{(N^\alpha n_\alpha)} = \frac{\sigma^{-1}}{\lambda_1} \tag{3.53}$$

$$T_\alpha T^\alpha = 0 = \eta^{-1}\alpha_T + T^1\sigma^{-1} \qquad \Longrightarrow T^1 = -\frac{\eta^{-1}}{\sigma^{-1}}\alpha_T \tag{3.54}$$

$$\begin{aligned} T_\alpha N^\alpha &= \eta^{-1} = T^a(N_\alpha E^\alpha_a) \\ &= -\eta^{-1} + T^A\lambda_A \end{aligned} \qquad \Longrightarrow T^A\lambda_A = 2\eta^{-1} \tag{3.55}$$

$$T_\alpha E^\alpha_A = 0 = \alpha_T(N_\alpha E^\alpha_A) + \overset{(0)}{g}_{AB} T^B \Longrightarrow \lambda_A = -\frac{\overset{(0)}{g}_{AB} T^B}{\alpha_T} \tag{3.56}$$

Now if we assume that $\eta^{-1} = \sigma^{-1} = \lambda_A = -1$ and $g_{AB} = p^{-2}\delta_{AB}$, we will find from (3.53) to (3.56)

$$\lambda_1 = -\frac{1}{p^{-2}}, \quad N_1 = -\frac{1}{p^{-2}}, \quad N_A = -1, \tag{3.57}$$

$$\alpha_T = p^{-2}, \qquad T^1 = -1, \qquad T^A = 1 \tag{3.58}$$

Now we can already calculate the q_{ab} tensor defined by (3.52). After simple calculations we find

$$q_{ab} = \begin{pmatrix} \frac{1}{p^{-2}} & 0 & 0 \\ 1 & p^{-2} & 0 \\ 1 & 0 & p^{-2} \end{pmatrix} \tag{3.59}$$

and using (3.51) we can finally find

$$\begin{aligned} X_1 &= \frac{1}{p^{-2}} X^1 \\ &= \frac{1}{p^{-2}} X^1_{(0)} - \frac{1}{2}\chi^{-1} u \left(\gamma_{1b} X^b_{(0)} + \frac{\gamma_{11} + \gamma_{12} + \gamma_{13}}{p^{-2}} X^1_{(0)} \right) \end{aligned} \tag{3.60}$$

$$\begin{aligned} X_A &= X^1 + p^{-2} X^A \\ &= X^1_{(0)} + p^{-2} X^A_{(0)} - \frac{1}{2}\chi^{-1} u \left(\gamma_{Ab} X^b_{(0)} + \frac{\gamma_{A1} + \gamma_{A2} + \gamma_{A3}}{p^{-2}} X^1_{(0)} \right) \end{aligned} \tag{3.61}$$

We can find g_*^{ab} using its defining equations (2.67) and (2.68) and the fact that g_{1a} and g_{AB} $(A \neq B)$ components of the metric are 0:

$$\frac{1}{p^{-2}} g_*^{a1} + g_*^{a2} + g_*^{a3} = 0, \quad g_*^{1c} g_{cA} = -p^{-2},$$
$$g_*^{Ac} g_{cA} = 1, \quad g_*^{Ac} g_{cB} = 0.$$

It follows from these relations that

$$g_*^{ab} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & p^2 & 0 \\ -1 & 0 & p^2 \end{pmatrix} \tag{3.62}$$

Using the defining equations of $\hat{\gamma}_{ab}$ (3.26) (the part of γ tensor which doesn't contribute to the stress-energy tensor, and is the source for gravitational waves), we can find that

$$\hat{\gamma}_{ab} = \begin{pmatrix} \hat{\gamma}_{22} & \hat{\gamma}_{23} \\ \hat{\gamma}_{23} & -\hat{\gamma}_{22} \end{pmatrix} \tag{3.63}$$

Now we can calculate the stress-energy tensor using (2.69).

$$\begin{aligned} 16\pi S^{ab} &= -(\gamma_{cd} g_*^{cd}) n^a n^b - (\gamma_{cd} n^c n^d) g_*^{ab} + (g_*^{ac} n^b n^d + g_*^{bc} n^a n^d)\gamma_{cd} \\ &= [2\gamma_{11} + p^2\gamma_{22} + p^2\gamma_{33} - 2\gamma_{12} - 2\gamma_{13}]\delta_1^a \delta_1^b \\ &\quad - \gamma_{11} g_*^{ab} + (g_*^{ac}\delta_1^b \delta_1^d + g_*^{bc}\delta_1^a\delta_1^d)\gamma_{cd} \end{aligned}$$

Thus

$$\begin{aligned} 16\pi S^{11} &= -p^2(\gamma_{22} + \gamma_{33}), \\ 16\pi S^{1B} &= p^2 \gamma_{1B}, \\ 16\pi S^{AB} &= -p^2 \gamma_{11} \delta_{AB}. \end{aligned}$$

We can see that this results are identical to the stress-energy tensor found in the case of timelike geodesics. Looking back at the equation (3.60), we can say that *the wave components of the signal do not contribute to* X^1, just like in the timelike case.

And also, just like in the timelike case, if the test particles are initially moving in the (2,3)-2-surface (which is also the signal front line), so that $X^1_{(0)} = \bar{V}_{(0)1} = 0$ and if $S^{B1} \neq 0$ then $X^1 \neq 0$. In the current case of lightlike geodesics we can find that $S^{1B} = Q^B$, where Q^a is the surface energy current, given by the equation (2.75). Hence, *if there is surface energy current in the lightlike shell and the particles are initially moving in the signal front, then* $X^1 \neq 0$, *so they are displaced out of the 2-surface after encountering the signal.* This statement is also true for the discussed case of timelike geodesic.

Now suppose there is no surface energy current and the test particles are initially on the signal front with no relative velocity in any direction ($S^{1B} = Q^B = 0$ and $X^1_{(0)} = \bar{V}_{(0)b} = 0$). This means that X^1 is 0 and for X^A we can write

$$p^{-2} X^A = p^{-2} X^A_{(0)} - \frac{1}{2}\chi^{-1}\, u \left(\gamma_{AB} X^B_{(0)}\right). \tag{3.64}$$

We can decompose γ_{AB} into $\hat{\gamma}_{AB}$ and $\bar{\gamma}_{AB}$ parts just like in (3.25), where according to the equation (3.27), $\bar{\gamma}_{AB}$ is given by

$$\bar{\gamma}_{AB} = \frac{1}{2} g_*^{cd} \gamma_{cd} g_{AB} + 2 \frac{1}{(n^\alpha N_\alpha)} \gamma_{(A} N_{B)} - \frac{\gamma_{cd} n^c n^d}{(n^\alpha N_\alpha)^2} (N_A N_B - \frac{1}{2} N^\alpha N_\alpha g_{AB}) \tag{3.65}$$

Before doing this decomposition we need to derive an equation for $\bar{\gamma}_{ab}$. For this task it is more convenient to chose new transversal vector N'^μ

$$N' \cdot N' = 0, \quad N' \cdot E_{(A)} = 0, \quad N' \cdot n = -1$$

From these definitions follows that

$$N'_a = -\delta^1_a.$$

From equations (2.67) and (2.68) we find $g_*'^{ab}$:

$$g_*'^{ab} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p^2 & 0 \\ 0 & 0 & p^2 \end{pmatrix}.$$

Because γ_{ab}, $\hat{\gamma}_{ab}$ and $\bar{\gamma}_{ab}$ are intrinsic tensors, they are independent of the choice of the transversal vector so using the obtained values for N'_a and $g_*'^{ab}$ we can find from the equation (3.65) and the expression for S^{11}

$$\bar{\gamma}_{AB} = \frac{1}{2} g_*'^{cd} \gamma_{cd} g_{AB} = \frac{1}{2} (\gamma_{22} + \gamma_{33}) \delta_{AB} = -8\pi S^{11} p^{-2} \delta_{AB}$$

Considering equation (3.64) and the obtained value for $\bar{\gamma}_{ab}$ we can write

$$\begin{aligned} X^A &= X^A_{(0)} - \frac{1}{2} \chi^{-1}\, u\, \left(p^2 (\hat{\gamma}_{AB} + \bar{\gamma}_{AB}) X^B_{(0)}\right) \\ &= X^A_{(0)} - \frac{1}{2} \chi^{-1}\, u\, \left(p^2 (\hat{\gamma}_{AB} - 8\pi S^{11} p^{-2} \delta_{AB}) X^B_{(0)}\right) \\ &= (1 + 4\pi\, \chi^{-1}\, u\, S^{11}) (\delta_{AB} - \frac{1}{2} \chi^{-1}\, u\, p^2 \hat{\gamma}_{AB}) X^B_{(0)}, \end{aligned} \tag{3.66}$$

where in the last step we omit the u^2 term. The factor $\delta_{AB} - \frac{1}{2}\chi^{-1}\, u\, p^2 \hat{\gamma}_{AB}$, as it depends on $\hat{\gamma}_{AB}$ describes the distortion effect of the wave part of the signal on the test particles. The other factor $1 + 4\pi\, \chi^{-1}\, u\, S^{11}$, like in the timelike geodesic case, is related to the lightlike shell and has focusing effect. So the conclusion is the same as in the corresponding case of the timelike geodesic: *if the lightlike shell has no surface energy current in parallel with a gravitational wave, the effect of the signal on the test particles at rest in the signal front is a displacement relative to each other and a distortion due to the gravitational wave diminished by the presence of the lightlike shell.*

4 Conclusion

All the discussions in the previous chapter were dedicated to the possible detection of the impulsive gravitational waves. The detection of these waves is very important as it can spread light on many problems related to the their sources. As it has already been mentioned in the introduction, different cataclysmic astrophysical events, like supernova or a collision of neutron stars, can be a source to such impulsive waves. These events usually take place in very short time scales, hence, in comparison with non-impulsive gravitational waves, impulsive gravitational waves last much shorter and don't result in oscillatory motion of the test particles. This means that current detectors, which are capable of registering non-impulsive gravitational waves, might not be very well suited for detecting impulsive gravitational waves. To overcome the detection problem either new types of detectors have to be designed or the existing detectors have to be modified.

The most sensitive gravitational wave detectors today are essentially laser interferometers. The most famous of them are the detectors of LIGO (Laser Interferometer Gravitational Wave Observatory), which in February 2016 announced the first direct detection of gravitational waves emitted from a system of spiraling and finally merging black holes. LIGO interferometers use Fabry-Pérot (FR) cavities on each arm, which store photons for a short time while they bounce between the mirrors. Gravitational waves cause changes in the distance between these mirrors which are detected by the interferometer.

As it has been explained, impulsive gravitational waves are not capable of creating such oscillations in the mirrors, but they rather result in one-time displacement of the mirrors. This kind of small displacements don't result in great change in the interference pattern of the laser beam. This means that a detector should have higher sensitivity to detect impulsive gravitational waves, compared to non-impulsive gravitational waves. The sensitivity of the detectors can be increased by finding more elastic materials for connecting the mirrors in the FR cavities, so that a small relative displacement of these mirrors is able to generate oscillations of the mirrors. Impulsive gravitational waves are in general created in more powerful events than non-impulsive gravitational waves. This means that there is more energy deposited in them, so in principle even further events can be detected by impulsive gravitational waves.

Further questions that can be discussed include the effects that a massless particle will have on an incident lightlike hypersurface. It will surely create some disturbance on the hypersurface, similar to the disturbances that a falling stone creates on a water surface.

Appendix A: Geodesic Deviation Vector

In this appendix we will discuss the geodesic deviation vector and important relations which have been used throughout the thesis.

Let's consider two geodesics labeled by y_0 and y_1 (fig. A.1). Each of them is described by a relation $y^\alpha(t)$, where t is an affine parameter. Now let's introduce a family of geodesics in the space between the curves y_0 and y_1. Let's also introduce a parameter $s = [0, 1]$ to distinguish each of these geodesics from the others. We will associate the curve y_0 with $s = 0$ and the curve y_1 with $s = 1$. Each of the other geodesic curves between y_0 and y_1 will be labeled by a distinct value of s.

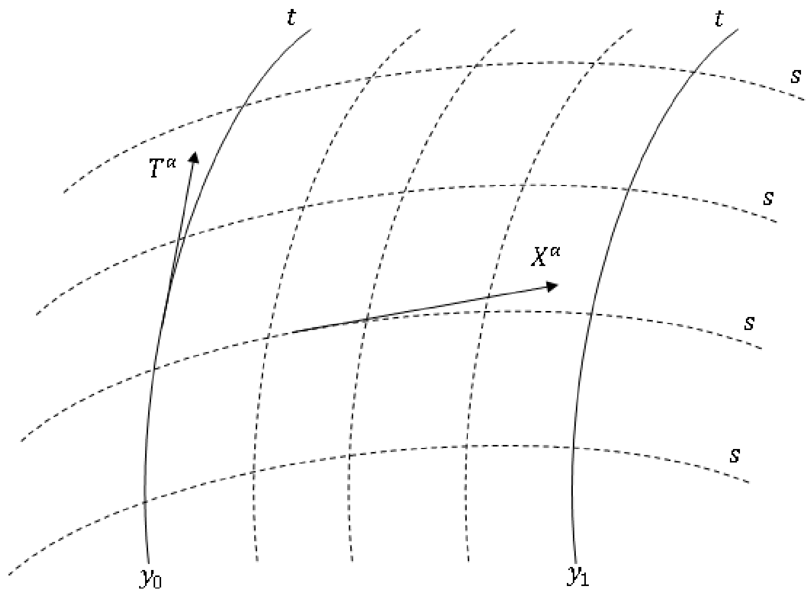

Figure A.1: Geodesic and geodesic deviation vectors of two neighboring particles.

We can collectively describe these family of geodesics by a relation $y^\alpha(s, t)$, where s specifies one geodesic curve and t is an affine parameter of that curve. It follows that if we keep t to some fixed value and vary s instead, we will obtain a new curve. Doing this procedure for each fixed value of t will yield a family of new curves, each labeled by a t and parameterized by s.

In this notation, the tangent vector field to the geodesics is given by $T^\alpha = \partial y^\alpha/\partial t$ and the tangent vector field to the t labeled curves is given by $X^\alpha = \partial y^\alpha/\partial s$. The family $X^\alpha|_{s=0}$ will have the geometrical meaning of the deviation vector between y_0 and y_1 curves.

As T^α is the tangent vector to the geodesic curve its parallel transport along the geodesic is 0.

$$\mathrm{D_T} T^\alpha = 0, \tag{A.1}$$

where $\mathrm{D_T}$ is in general the covariant derivative along the tangent vector T^α (in our case geodesic curve). It is clear that the following relation holds

$$\frac{\partial^2 y^\alpha}{\partial t \partial s} = \frac{\partial^2 y^\alpha}{\partial s \partial t}$$

In the covariant form it will transform into

$$\mathrm{D_T} X^\alpha = \mathrm{D_X} T^\alpha \tag{A.2}$$

Using equations (A.1) and (A.2) and also the fact that $T^\alpha X_\alpha$ is a scalar we can write

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(T^\alpha X_\alpha) &= \mathrm{D_T}(T^\alpha X_\alpha) = X_\alpha \, \mathrm{D_T} T^\alpha + T^\alpha \, \mathrm{D_T} X_\alpha \\ &= T^\alpha \, \mathrm{D_T} X_\alpha = T^\alpha \, \mathrm{D_X} T_\alpha = \frac{1}{2} \mathrm{D_X}(T^\alpha T_\alpha) \\ &= 0 \end{aligned} \tag{A.3}$$

So, the product of the geodesic vector and the geodesic deviation vector is constant along y_0.
Another very important relation is the equation of the acceleration of the geodesic deviation vector along y_0:

$$\mathrm{D_T}^2 X^\alpha = \mathrm{D_T}(\mathrm{D_T} X^\alpha)$$

In the flat space-time this acceleration is clearly 0, because although two particles can have relative velocity, they cannot have relative acceleration. This gives us a hint to assume that the value of $\mathrm{D_T}^2 X^\alpha$ must contain some indication about the curvature of the space-time. And in fact detailed calculations show that this acceleration is proportional to the Riemann curvature tensor:

$$\mathrm{D_T}^2 X^\alpha = -R^\alpha{}_{\beta\gamma\delta} T^\beta X^\gamma T^\delta. \tag{A.4}$$

This equation is the cornerstone of our calculations in the chapter 3.

Appendix B: Projection of The Deviation Vector

In this appendix we will show the detailed derivation of the projection of the deviation vector X^μ in the directions of E^μ_a, which has been suggested in [5]. The geodesic vectors of the particles are timelike. The lightlike case is discussed in the section 3.3. The notation are the same as in the section 3.1. The projections X_a in the directions of E^μ_a are given by

$$X_a = g_{\mu\nu} X^\mu E^\nu. \tag{B.1}$$

In the section 3.1 equations were derived for the jump vectors P^μ, W^μ and F^μ_a of the vectors T^μ, X^μ and E^μ_a accordingly:

$$P^\mu = -\gamma^\mu_\nu T^\nu_{(0)} + n^\mu_{(0)} \frac{(\gamma_{\nu\rho} T^\nu_{(0)} T^\rho_{(0)})}{2(T^\nu_{(0)} n_{(0)\nu})}, \tag{B.2}$$

$$W^\mu = X_{(0)} P^\mu \tag{B.3}$$

$$F^\mu_a = -\frac{1}{2}\gamma^\mu_\lambda e^\lambda_a + n^\mu_{(0)} \frac{(\gamma_{\mu\nu} e^\mu_a T^\nu_{(0)})}{2(T^\nu_{(0)} n_{(0)\nu})}. \tag{B.4}$$

Using (3.7), for small positive u we can write

$$\begin{aligned}
[X^\mu{}_{,\lambda}] &= X^\mu{}_{,\lambda}|_+ - X^\mu{}_{,\lambda}|_- \\
&= X^\mu{}_{,u} u_{,\lambda}|_+ - X^\mu{}_{,u} u_{,\lambda}|_- = u_{,\lambda}(X^\mu{}_{,u}|_+ - X^\mu{}_{,u}|_-) \\
&= u_{,\lambda}\left(\frac{X^\mu - X^\mu_{(0)}}{u} - \frac{\bar{X}^\mu - X^\mu_{(0)}}{u}\right) = u_{,\lambda}\frac{X^\mu - \bar{X}^\mu}{u} \\
\Longrightarrow \eta u W^\mu n_\lambda &= u_{,\lambda}(X^\mu - \bar{X}^\mu) \\
\Longrightarrow \eta \chi^{-1} u W^\mu n_\lambda &= n_\lambda (X^\mu - \bar{X}^\mu)
\end{aligned}$$

and finally

$$X^\mu = \bar{X}^\mu + \eta\chi^{-1} u W^\mu. \tag{B.5}$$

where a bar on $\bar{X}^\mu$ represents the same tensor if no impulsive signal was present on the hypersurface (in other words the whole space-time is covered by the metric of $\mathcal{M}^-$ domain). If we include also the cases of $u = 0$ and $u < 0$ we can write

$$X^\mu = \bar{X}^\mu + \eta\chi^{-1} u\theta(u) W^\mu. \tag{B.6}$$

Similar expressions can be derived for $g_{\mu\nu}$ tensor and E^μ_a vector:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \eta\,\chi^{-1}\,u\,\gamma_{\mu\nu} \tag{B.7}$$

$$E^\mu_a = \bar{E}^\mu_a + \eta\,\chi^{-1}\,u\,F^\mu_a \tag{B.8}$$

We are only interested in the linear approximation of the projection X_a. Plugging equations (B.5), (B.7) and (B.8) into equation (B.1), we will find

$$\begin{aligned} X_a &= g_{\mu\nu} X^\mu E^\nu_a \\ &= (\bar{g}_{\mu\nu} + \eta\,\chi^{-1}\,u\,\gamma^{\mu\nu})(\bar{X}^\mu + \eta\,\chi^{-1}\,u\,W^\mu)(\bar{E}^\nu_a + \eta\,\chi^{-1}\,u\,F^\nu_a) \\ &= \bar{g}_{\mu\nu}\bar{X}^\mu \bar{E}^\nu_a + \eta\,\chi^{-1}\,u\,(\gamma_{\mu\nu}\bar{X}^\mu\bar{E}^\nu_a + \bar{g}_{\mu\nu}W^\mu\bar{E}^\nu_a + \bar{g}_{\mu\nu}\bar{X}^\mu F^\nu_a) \\ &= \bar{g}_{\mu\nu}\bar{X}^\mu \bar{E}^\nu_a + \eta\,\chi^{-1}\,u\,(\gamma_{\mu\nu}X^\mu_{(0)}e^\nu_a + g^{(0)}_{\mu\nu}W^\mu e^\nu_a + g^{(0)}_{\mu\nu}X^\mu_{(0)}F^\nu_a) \end{aligned} \tag{B.9}$$

On the last step we have decomposed $\bar{g}_{\mu\nu}$, $\bar{X}^\mu$ and E^ν_a into series around the hypersurface and kept only the terms independent of u as we have demanded X_a to be linear on u. The linear term in parenthesis can be modified:

$$\begin{aligned} (...) = \gamma_{\mu\nu}X^\mu_{(0)}e^\nu_a + g^{(0)}_{\mu\nu}e^\nu_a X_0 \left(-\gamma^\mu_\rho T^\rho_{(0)} + n^\mu_{(0)} \frac{(\gamma_{\rho\lambda}T^\rho_{(0)}T^\lambda_{(0)})}{2(T^\alpha_{(0)}n_{(0)\alpha})} \right) \\ + g^{(0)}_{\mu\nu}X^\mu_{(0)} \left(-\frac{1}{2}\gamma^\nu_\rho e^\rho_a + n^\nu_{(0)} \frac{(\gamma_{\rho\lambda}e^\rho_a T^\lambda_{(0)})}{2(T^\alpha_{(0)}n_{(0)\alpha})} \right) \\ = \frac{1}{2}\gamma_{\mu\nu}X^\mu_{(0)}e^\nu_a - \frac{1}{2}X_0(\gamma_{\rho\lambda}e^\rho_a T^\lambda_{(0)}) = \frac{1}{2}\gamma_{\mu\nu}e^\nu_a\left(X^\mu_{(0)} - X_0 T^\mu_{(0)}\right) \\ = \frac{1}{2}X^b_{(0)}\gamma_{\mu\nu}e^\nu_a e^\mu_b = \frac{1}{2}X^b_{(0)}\gamma_{ab} \end{aligned} \tag{B.10}$$

As we take X_a to be linear on u, the first term in X_a (the term with bar tensors) has to be taken with linear approximation.

$$\begin{aligned} \bar{g}_{\mu\nu}\bar{X}^\mu\bar{E}^\nu_a &= g^{(0)}_{\mu\nu}X^\mu_{(0)}e^\nu_a + u\,\left.\frac{\mathrm{d}\bar{X}_a}{\mathrm{d}u}\right|_{u=0} \\ &= X_0(T_{(0)\alpha}e^\alpha_a) + X^b_{(0)}g^{(0)}_{\mu\nu}e^\mu_b e^\nu_a + u\,\left.\frac{\mathrm{d}\bar{X}_a}{\mathrm{d}u}\right|_{u=0} \\ &= X^b_{(0)}(T_{(0)\beta}e^\beta_b)(T_{(0)\alpha}e^\alpha_a) + X^b_{(0)}g^{(0)}_{ab} + u\,\left.\frac{\mathrm{d}\bar{X}_a}{\mathrm{d}u}\right|_{u=0} \end{aligned} \tag{B.11}$$

Putting equations (C.10), (B.10) and (B.11) together will finally yield

$$X_a = \left(p_{ab} + \frac{1}{2}\eta\chi^{-1}u\gamma_{ab}\right)X^b_{(0)} + u\,\bar{V}_{(0)a},$$

where p_{ab} is

$$p_{ab} = g_{ab} + (T_{(0)\mu}e^\mu_a)(T_{(0)\nu}e^\nu_b), \tag{B.12}$$

$\bar{V}_{(0)a}$ is given by

$$\bar{V}_{(0)a} = \frac{\mathrm{d}\bar{X}_a}{\mathrm{d}u}$$

and is evaluated at $u = 0$.

Appendix C: Parallel Transported Vectors

In this appendix we are discussing a particle crossing a lightlike hypersurface which divides the space-time into two domains $\mathcal{M}^{\pm}$. We will assume the hypersurface is described by an equation $u = 0$, and $u < 0$ corresponds to $\mathcal{M}^-$ domain and $u > 0$ corresponds to $\mathcal{M}^+$ domain. Particle is moving from $\mathcal{M}^-$ domain to $\mathcal{M}^+$ domain and the geodesic vector T^μ along with other vectors parallel transported along the geodesic curve can be parameterized by u.

We will show that the product of any two vectors U^μ and V^μ, which have been parallel transported along the geodesic curve, does not suffer a jump and does not contain linear or higher terms in u. This result is true for all types of geodesic vectors - null, timelike and spacelike. We will denote the derivative along the geodesic curve by D_T. So we can write

$$D_T U^\mu = D_T V^\mu = 0, \quad (T^\mu T_\mu) = \epsilon, \quad \epsilon = 0, \pm 1. \tag{C.1}$$

We will denote the jump vectors of U^μ and V^μ by Y^μ and Z^μ respectively and the jump tensor of the metric by $\gamma_{\mu\nu}$ as before:

$$[U^\mu_{,\lambda}] = \eta Y^\mu n_\lambda, \tag{C.2}$$

$$[V^\mu_{,\lambda}] = \eta Z^\mu n_\lambda, \tag{C.3}$$

$$[g_{\mu\nu,\lambda}] = \eta \gamma_{\mu\nu} n_\lambda \tag{C.4}$$

where η is some normalization factor and $n_\lambda = \chi^{-1}\partial_\lambda u$ is the normal vector to the hypersurface, with χ^{-1} normalisation factor.

$$\begin{aligned}
[D_T U^\mu] = 0 = \left[U^\mu{}_{|\nu} T^\nu\right] &= T^\nu_{(0)}[\partial_\nu U^\mu + \Gamma^\mu_{\nu\rho} U^\rho] = T^\nu_{(0)}\left([U^\mu{}_{,\nu}] + U^\rho_{(0)}[\Gamma^\mu_{\nu\rho}]\right) \\
&= \eta\left(Y^\mu (T^\nu_{(0)} n_\nu) + T^\nu_{(0)} U^\rho_{(0)}\left(\gamma^\mu_{(\nu} n_{\rho)} - \frac{1}{2}\gamma_{\nu\rho} n^\mu\right)\right) \\
&= \eta\left(Y^\mu (T^\nu_{(0)} n_\nu) + \frac{1}{2} T^\nu_{(0)} U^\rho_{(0)}\left(\gamma^\mu_\nu n_\rho + \gamma^\mu_\rho n_\nu - \gamma_{\nu\rho} n^\mu\right)\right)
\end{aligned}$$

It follows that

$$Y^\mu = -\frac{1}{2(T^\nu_{(0)} n_\nu)} U^\rho_{(0)} T^\nu_{(0)}\left(\gamma^\mu_\nu n_\rho + \gamma^\mu_\rho n_\nu - \gamma_{\nu\rho} n^\mu\right) \tag{C.5}$$

A similar result can be obtained for Z^μ:

$$Z^\mu = -\frac{1}{2(T^\nu_{(0)} n_\nu)} V^\rho_{(0)} T^\nu_{(0)}\left(\gamma^\mu_\nu n_\rho + \gamma^\mu_\rho n_\nu - \gamma_{\nu\rho} n^\mu\right) \tag{C.6}$$

Following the same steps as in the derivation of the equation (B.5), we can write

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \eta\,\chi^{-1}\,u\,\gamma_{\mu\nu}\,, \tag{C.7}$$
$$U^{\mu} = \bar{U}^{\mu} + \eta\,\chi^{-1}\,u\,Y^{\mu}\,, \tag{C.8}$$
$$\underset{a}{V^{\mu}} = \bar{V}^{\mu} + \eta\,\chi^{-1}\,u\,Z^{\mu}\,, \tag{C.9}$$

Where the bar terms represent the corresponding tensors in the absence of a singular hypersurface, in other words if the $\mathcal{M}^{-}$ covered the whole spacetime. Using equation (C.7), (C.8) and (C.9), we can write for the linear approximation of the product $g_{\mu\nu}U^{\mu}V^{\nu}$:

$$\begin{aligned} g_{\mu\nu}U^{\mu}V^{\nu} &= (\bar{g}_{\mu\nu} + \eta\,\chi^{-1}\,u\,\gamma^{\mu\nu})(\bar{U}^{\mu} + \eta\,\chi^{-1}\,u\,Y^{\mu})(\bar{V}^{\nu} + \eta\,\chi^{-1}\,u\,Z^{\nu}) \\ &= \bar{g}_{\mu\nu}\bar{U}^{\mu}\bar{V}^{\nu} + \eta\,\chi^{-1}u\,(\gamma_{\mu\nu}\bar{U}^{\mu}\bar{V}^{\nu} + \bar{g}_{\mu\nu}Y^{\mu}\bar{V}^{\nu} + \bar{g}_{\mu\nu}\bar{Z}^{\nu}U^{\mu}) \\ &= \bar{g}_{\mu\nu}\bar{U}^{\mu}\bar{V}^{\nu} + \eta\,\chi^{-1}u\,(\gamma_{\mu\nu}U^{\mu}_{(0)}V^{\nu}_{(0)} + g^{(0)}_{\mu\nu}Y^{\mu}V^{\nu}_{(0)} + g^{(0)}_{\mu\nu}Z^{\nu}U^{\mu}_{(0)}) \end{aligned} \tag{C.10}$$

On the last step we have decomposed $\bar{g}_{\mu\nu}$, $\bar{U}^{\mu}$ and V^{ν} into series around the hypersurface and kept only the terms independent of u as we have demanded the product to be linear on u. The term in parenthesis clearly arises because of the jumps that the derivatives of the vectors and the metric suffer. We will use the equations (C.5) and (C.6) to modify this term:

$$\begin{aligned} (...) &= \gamma_{\mu\nu}U^{\mu}_{(0)}V^{\nu}_{(0)} + Y_{\nu}V^{\nu}_{(0)} + Z_{\mu}U^{\mu}_{(0)} \\ &= \gamma_{\mu\nu}U^{\mu}_{(0)}V^{\nu}_{(0)} + \left(-\frac{1}{2(T^{\alpha}_{(0)}n_{\alpha})}U^{\rho}_{(0)}T^{\sigma}_{(0)}\left(\gamma_{\nu\sigma}n_{\rho} + \gamma_{\nu\rho}n_{\sigma} - \gamma_{\sigma\rho}n_{\nu}\right)\right)V^{\nu}_{(0)} \\ &\qquad + \left(-\frac{1}{2(T^{\alpha}_{(0)}n_{\alpha})}V^{\rho}_{(0)}T^{\sigma}_{(0)}\left(\gamma_{\nu\sigma}n_{\rho} + \gamma_{\nu\rho}n_{\sigma} - \gamma_{\sigma\rho}n_{\nu}\right)\right)U^{\nu}_{(0)} \\ &= \gamma_{\mu\nu}U^{\mu}_{(0)}V^{\nu}_{(0)} - \frac{1}{2(T^{\alpha}_{(0)}n_{\alpha})}T^{\sigma}_{(0)}(U^{\rho}V^{\nu} + V^{\rho}U^{\nu})\left(\gamma_{\nu\sigma}n_{\rho} + \gamma_{\nu\rho}n_{\sigma} - \gamma_{\sigma\rho}n_{\nu}\right) \\ &= \gamma_{\mu\nu}U^{\mu}_{(0)}V^{\nu}_{(0)} - \frac{1}{2}\gamma_{\nu\rho}(U^{\rho}V^{\nu} + V^{\rho}U^{\nu}) = 0 \end{aligned}$$

So we are left with

$$g_{\mu\nu}U^{\mu}V^{\nu} = \bar{g}_{\mu\nu}\bar{U}^{\mu}\bar{V}^{\nu}.$$

We can expand the left side of this relation into a series until the linear term:

$$g_{\mu\nu}U^{\mu}V^{\nu} = g^{(0)}_{\mu\nu}U^{\mu}_{(0)}V^{\nu}_{(0)} + u\frac{\mathrm{d}}{\mathrm{d}u}(g_{\mu\nu}U^{\mu}V^{\nu})$$

As the product $(g_{\mu\nu}U^{\mu}V^{\nu})$ is a scalar quantity we can write

$$\frac{\mathrm{d}}{\mathrm{d}u}(g_{\mu\nu}U^{\mu}V^{\nu}) = D_T(g_{\mu\nu}U^{\mu}V^{\nu}) = g_{\mu\nu}(V^{\nu}D_TU^{\mu} + U^{\mu}D_TV^{\nu}) = 0$$

So,

$$g_{\mu\nu}U^{\mu}V^{\nu} = g^{(0)}_{\mu\nu}U^{\mu}_{(0)}V^{\nu}_{(0)}.$$

Thus we have proved the claim we have made in the beginning of this appendix, saying that *the product of any two vectors parallel transported* *along the geodesic curve will not suffer a jump and will not contain u-terms. In other words, such a product will conserve it's value along the whole geodesic curve and, particularly, will be equal to its value on the singular hypersurface.*

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